PAIRS OF LINES

VECTORS 2

INU0114/514 (MATHS 1)

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Objectives

The purpose of this presentation is to cover the following topics:

- Consistency of equations.
- Be able to show whether two vector lines are parallel or not.
- If two vector lines are not parallel then be able to show whether the lines intersect or not.

We'll also practice using both column vector and **ijk** notations in these problems.

Consistency of equations

Consider the equations

$$\begin{aligned} 3\lambda - 2\mu &= 10 \\ \lambda + \mu &= 0 \\ -4\lambda + 3\mu &= -14 \end{aligned}$$

Can you find values of μ and λ which satisfy all three equations?

Choose any pair and solve to get $\mu = 2$ and $\lambda = -2$. Substitute into the other equation. You should find these values give $LHS = RHS$. Therefore the equations are consistent; the values are true in all 3 equations.

Now consider the equations

$$\begin{aligned} 5\lambda - \mu &= 1 \\ -5\lambda + 2\mu &= 4 \\ 2\lambda - 3\mu &= 1 \end{aligned}$$

Can you find values of μ and λ which satisfy all three equations?

Choose any pair and solve; e.g. the first two, to get $\mu = 5$ and $\lambda = \frac{6}{5}$. Substitute into the other equation to find $-\frac{63}{5} \neq 1$. Your values don't work! Therefore the equations are inconsistent. There are no values for μ or λ.

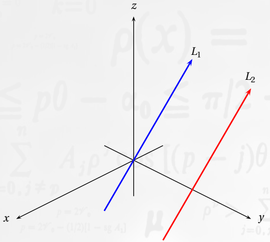
Pairs of lines

In 2D (Euclidean) geometry a pair of straight lines are either parallel or, if not, they will intersect somewhere.

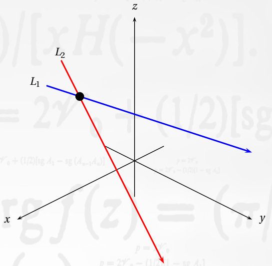
Pairs of lines in 3D space do not necessarily intersect when they aren't parallel. The location of two lines in 3D space may such that

1. They are parallel and distinct (no intersection)
2. They are parallel and coincident (infinite intersections)
3. They are not parallel and intersect at one point.
4. They are not parallel and don't intersect.
   We call them *skew lines* in this case.

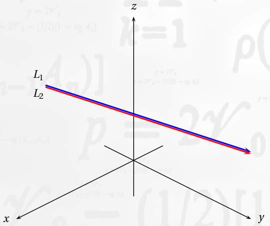
These possibilities are shown on the following slide.



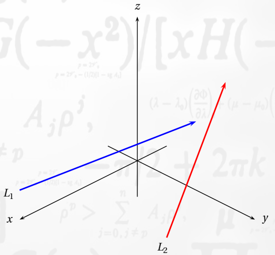
Parallel and distinct; no intersections.



Not parallel; one intersection.



Parallel and coincident; infinite intersections.



Not parallel; no intersection (skew).

Parallel lines

Recall that two vectors are parallel if they satisfy $\mathbf{b} = k\mathbf{a}$ where k is a constant.

Consider the two lines

$$\begin{aligned} \mathbf{r_1} &= \mathbf{a_1} + \lambda \mathbf{d_1} \\ \mathbf{r_2} &= \mathbf{a_2} + \mu \mathbf{d_2} \end{aligned}$$

To see if the lines are parallel we just need to check the direction vectors $\mathbf{d_1}$ and $\mathbf{d_2}$.

If the lines are parallel then we would expect to find

$$\mathbf{d_2} = k\mathbf{d_1}$$

If lines are parallel they could be distinct (no intersection) or coincident (infinite intersections).

Pairs of lines

Show that the lines given by

$$\mathbf{r_1} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ -9 \end{pmatrix} \quad \text{and} \quad \mathbf{r_2} = \begin{pmatrix} 10 \\ -2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ -18 \end{pmatrix}$$

are parallel.

The direction vectors are

$$\mathbf{d_1} = \begin{pmatrix} -1 \\ 2 \\ -9 \end{pmatrix} \quad \text{and} \quad \mathbf{d_2} = \begin{pmatrix} -2 \\ 4 \\ -18 \end{pmatrix}$$

Comparing the components we see: $\mathbf{d_2} = 2\mathbf{d_1}$.

The lines are parallel.

Pairs of lines

Consider the lines

$$\mathbf{r_1} = \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{r_2} = \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix}$$

Are they parallel?

The direction vectors are

$$\mathbf{d_1} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{d_2} = \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix}$$

The x and y components of $\mathbf{d_2}$ are a factor of -2 larger than $\mathbf{d_1}$. But the z component is not. Therefore $\mathbf{d_2} \neq k\mathbf{d_1}$.

The lines are not parallel.

Non-parallel lines

Consider the two lines

$$\mathbf{r_1} = \mathbf{a_1} + \lambda \mathbf{d_1}$$
$$\mathbf{r_2} = \mathbf{a_2} + \mu \mathbf{d_2}$$

If the lines are not parallel then can try to find an intersection.

If the lines intersect then there must be unique values of λ and μ such that

$$\mathbf{r_1} = \mathbf{r_2}$$

i.e. we should be able to solve the equations to find values of λ and μ at the intersection.

If no values of values of λ and μ can be found then the lines do not intersect and are said to be skew.

Pairs of lines

Consider the lines whose equations are

$$\mathbf{r_1} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and } \mathbf{r_2} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

Show that the lines are non-parallel and intersect.

The direction vectors are

$$\mathbf{d_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{d_2} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

The lines are not parallel because $\mathbf{d_1} \neq k\mathbf{d_2}$.

Let's see if the lines intersect...

If there is an intersection point (x, y, z) then we should be able to find it by solving $\mathbf{r_1} = \mathbf{r_2}$.

In parametric form the lines are:

$$\mathbf{r_1} = \begin{pmatrix} 1+\lambda \\ -1-\lambda \\ 3+\lambda \end{pmatrix} \quad \text{and } \mathbf{r_2} = \begin{pmatrix} 2+2\mu \\ 4+\mu \\ 6+3\mu \end{pmatrix}$$

Equating the components on each side gives three equations:

$$\begin{aligned} 1+\lambda &= 2+2\mu \\ -1-\lambda &= 4+\mu \\ 3+\lambda &= 6+3\mu \end{aligned}$$

We have three equations and with two unknowns. We'll need to solve two and check for consistency...

$$\begin{aligned} 1+\lambda &= 2+2\mu \quad (1) \\ -1-\lambda &= 4+\mu \quad (2) \\ 3+\lambda &= 6+3\mu \quad (3) \end{aligned}$$

We can choose any pair of equations to solve — we'll always be able to find values of λ and μ.

Using equations (1) and (2)

$$\begin{aligned} 1+\lambda &= 2+2\mu \\ -1-\lambda &= 4+\mu \end{aligned}$$

Solve these simultaneously. In this case we find

$$\lambda = -3 \text{ and } \mu = -2$$

We solved a pair of equations to find $\lambda = -3$ and $\mu = -2$.

Are these consistent with equation (3) ? If they are, then the lines intersect. If not, then the lines are skew.

Substitute into (3):

$$\begin{aligned} LHS = 3 + (-3) &= 0 \\ RHS = 6 + 3(-2) &= 0 \end{aligned}$$

$LHS = RHS$. Since the values are consistent for all 3 equations then the intersect.

To find the point — substitute λ or μ into one of the line equations. For example:

$$\mathbf{r_1} = \begin{pmatrix} 1+\lambda \\ -1-\lambda \\ 3+\lambda \end{pmatrix} \qquad \therefore \mathbf{r_1} = \begin{pmatrix} 1-3 \\ -1-(-3) \\ 3-3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$$

The lines intersect at the point $(-2, 2, 0)$.

Pairs of lines

Consider the lines whose equations are

$$\begin{aligned}\mathbf{r_1} &= \mathbf{i}+\mathbf{k}+\lambda(\mathbf{i}+3\mathbf{j}+4\mathbf{k}) \\ \mathbf{r_2} &= 2\mathbf{i}+3\mathbf{j}+\mu(4\mathbf{i}-\mathbf{j}+\mathbf{k})\end{aligned}$$

Determine whether or not the lines intersect.

The direction vectors:

$$\mathbf{d_1}=\mathbf{i}+3\mathbf{j}+4\mathbf{k} \qquad \text{and} \qquad \mathbf{d_2}=\mathbf{4i}-\mathbf{j}+\mathbf{k}$$

Since $\mathbf{d_2} \neq k\mathbf{d_1}$ then the lines are not parallel.

Now, look for an intersection...

If there is an intersection point (x,y,z) then we should be able to find it by solving $\mathbf{r_1} = \mathbf{r_2}$.

$$\mathbf{i}+\mathbf{k}+\lambda(\mathbf{i}+3\mathbf{j}+4\mathbf{k}) = 2\mathbf{i}+3\mathbf{j}+\mu(4\mathbf{i}-\mathbf{j}+\mathbf{k})$$

Equating the components on each side leads to the equations

$$1+\lambda = 2+4\mu \quad (4)$$
$$3\lambda = 3-\mu \quad (5)$$
$$1+4\lambda = \mu \quad (6)$$

As before: choose two equations to solve for μ and λ then check consistency with the third equation.

Using (4) and (5):

$$1+\lambda = 2+4\mu$$
$$3\lambda = 3-\mu$$

Solving these gives $\mu = 0$ and $\lambda = 1$.

We just found $\mu = 0$ and $\lambda = 1$ — but are they consistent with the other equation (6)?

Substitute λ and μ into (6)

$$\begin{aligned} LHS &= 1 + 4(1) = 5 \\ RHS &= 0 \end{aligned}$$

Since $LHS \neq RHS$ then our values of $\mu = 0$ and $\lambda = 1$ are not consistent.

The lines do not intersect; they are skew lines.

Coincident lines

Consider the parallel lines

$$\mathbf{r}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \text{ and } \mathbf{r}_2 = \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 6 \\ 2 \end{pmatrix}$$

The lines are parallel because $\mathbf{d}_2 = 2\mathbf{d}_1$.

Now we'll check whether the two lines are *distinct* or *coincident.*

If the lines are coincident, then the point $(0,6,3)$ will also be on the line $\mathbf{r}_1$.

We should be able to find a value for λ that corresponds to this point.

The parametric equations for the lines are

$$\begin{aligned} 2-\lambda &= 0 \\ 3\lambda &= 6 \\ 1+\lambda &= 3 \end{aligned}$$

Solving the first equation gives $\lambda = 2$.

By inspection, we see that this value is also consistent with the other two equations.

Since the lines are parallel and pass through the shared point, then they are coincident lines.

(Note that we could have proved the same result using μ; by showing that $(2,0,1)$ is on $\mathbf{r}_2$).

Consider the parallel lines

$$\mathbf{r}_1 = \begin{pmatrix} 8 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} \text{ and } \mathbf{r}_2 = \begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ -6 \\ -18 \end{pmatrix}$$

The lines are parallel because $\mathbf{d}_2 = -3\mathbf{d}_1$.

Now we'll check whether the two lines are *distinct* or *coincident*.

If the lines are coincident, then the point $(-2, -3, -4)$ will also be on the line $\mathbf{r}_1$.

Can we find a value for λ that corresponds to this point?

The parametric equations for the lines are

$$\begin{aligned} 8+\lambda &= 2 \\ -3+2\lambda &= -3 \\ 2+6\lambda &= -4 \end{aligned}$$

Solving the first equation gives $\lambda = -6$.

However, this value fails in the other two equations.

The point $(-2,-3,-4)$ is only on $\mathbf{r_2}$.

The lines are parallel and distinct.

Test yourself

If you've read and understood the examples in these notes, you should be able to answer the following questions.

Decide if the following pairs of lines are parallel or not. If not, are they intersecting or skew?

1. $\mathbf{r} = -2\mathbf{i} + \mathbf{j} + \lambda(2\mathbf{i} - \mathbf{j})$ and $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mu(8\mathbf{i} - 4\mathbf{j})$
2. $\mathbf{r} = 2\mathbf{i} - \mathbf{k} + \lambda(-2\mathbf{i} + 4\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} - 5\mathbf{k} + \mu(\mathbf{i} - 3\mathbf{j} - \mathbf{k})$
3. $\mathbf{r} = 5\mathbf{i} - 4\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} - \mathbf{k})$
4. $\mathbf{r} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 6\mathbf{i} - 2\mathbf{j} + 8\mathbf{k} + \mu(-\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

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1. Parallel and distinct.
2. Not parallel; intersection at $(0, 4, -2)$.
3. Not parallel; skew.
4. Parallel and coincident.