VECTORS EQUATION OF A LINE

VECTORS 2

INU0114/514 (MATHS 1)

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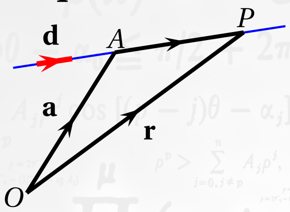
Objectives

The purpose of this presentation is to cover the following topics:

- Understand how straight lines (2D and 3D) can be expressed using vectors.
- Be able to find the vector equation of a line given two points on the line.
- Be able to write down equivalent parametric equations for a vector equation.
- Be able to write down a Cartesian equation from the parametric equations (lines in 2D)
- Be able to write down a symmetric equations from the parametric equations (lines in 3D)

Vector equation of a line



Suppose we have a line which passes through the point A.

The point P is a general point on the line. The position vector of A is $\mathbf{a}$.

The position vector of P is $\mathbf{r}$.

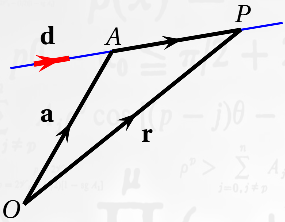
The vector equation linking these points is:

$$\mathbf{r} = \mathbf{a} + \overrightarrow{AP}$$

But $\overrightarrow{AP}$ is simply a multiple of vector $\mathbf{d}$ so that $\overrightarrow{AP} = \lambda\mathbf{d}$ instead.
The vector equation of the point P is given by:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$$

Where λ is a scalar (constant).



$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$$

The point P can be put anywhere along the line by varying the value of λ.

That means the equation represents all possible points on the line. It is therefore usually referred to as the *vector equation of the line.*

The vector **a** is the position vector of a given point on the line.

The vector **d** is called the direction vector; it specifies the direction of the line.

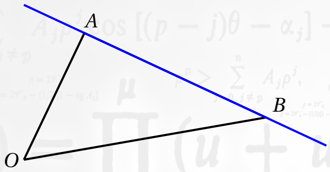
For example, the line

$$\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \lambda(2\mathbf{i} - 5\mathbf{j})$$

passes through the point $(3,4)$ and is parallel to vector $2\mathbf{i} - 5\mathbf{j}$.

Finding the vector equation of a line

Find the vector equation for the line through the points $A(3,6)$ and $B(6,2)$.



Draw a picture if necessary.
Accuracy doesn't matter!
To find the vector equation we need:

- The equation of the vector $\overrightarrow{AB}$.
- One point on the line AB

In the vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ the vector $\overrightarrow{AB}$ is the same as $\mathbf{d}$.

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (6\mathbf{i}+2\mathbf{j}) - (3\mathbf{i}+6\mathbf{j}) \\ &= 3\mathbf{i} - 4\mathbf{j}\end{aligned}$$

We can choose A or B for vector $\mathbf{a}$.

Using point A the vector equation is

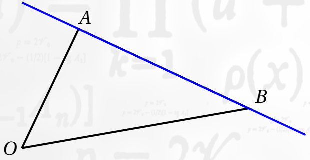
$$\mathbf{r} = 3\mathbf{i} + 6\mathbf{j} + \lambda(3\mathbf{i} - 4\mathbf{j})$$

Vector equation of a line (3D)

Find the vector equation for the line through the points $A(3,4,-7)$ and $B(1,-1,6)$.

We are looking to write the equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$.

Here is the picture to help:



The vector $\overrightarrow{AB}$ is the same as $\mathbf{d}$.

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (\mathbf{i} - \mathbf{j} + 6\mathbf{k}) - (3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}) \\ &= -2\mathbf{i} - 5\mathbf{j} + 13\mathbf{k} \end{aligned}$$

Using the point A to give $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$ then

$$\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k} + \lambda(-2\mathbf{i} - 5\mathbf{j} + 13\mathbf{k})$$

Vector equation to parametric equations

A vector equation can be converted to parametric equations if necessary.

Consider the vector equation of the line

$$\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \lambda(2\mathbf{i} - 6\mathbf{j})$$

Factorise the equation in terms of **i** and **j**

$$\mathbf{r} = (3 + 2\lambda)\mathbf{i} + (4 - 6\lambda)\mathbf{j}$$

The coefficients of **i** and **j** represent the x and y values of points on the line.

Therefore

$$x = 3 + 2\lambda, \, y = 4 - 6\lambda$$

These are parametric equations. The values of x and y depend on a parameter λ.

Vector equation to Cartesian equation

The Cartesian form of the vector equation is obtained by eliminating the parameter from the parametric equations.

Cartesian equation (2D)

Find the Cartesian equation for

$$\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \lambda(2\mathbf{i} - 6\mathbf{j})$$

The parametric equations are

$$x = 3 + 2\lambda,\ y = 4 - 6\lambda$$

Make λ the subject of one of those equations, e.g., $\lambda = \dfrac{x-3}{2}$.

Substitute into the other equation to get:

$$y = 4 - 6\left(\frac{x-3}{2}\right)$$

Rearrange and simplify to get:

$$y = 13 - 3x$$

Vector equation to symmetric equations

Lines in 3D space have a more complicated form of “Cartesian equation”; this time we end with symmetric equations involving x, y and z.

Symmetric equations (3D)

Find the parametric and Cartesian equations for the line

$$\mathbf{r} = 5\mathbf{i} + 2\mathbf{k} + \lambda(4\mathbf{i} - \mathbf{j} + 4\mathbf{k})$$

Factorising the **i**, **j** and **k** components:

$$\mathbf{r} = (5 + 4\lambda)\mathbf{i} - \lambda\mathbf{j} + (2 + 4\lambda)\mathbf{k}$$

The parametric equations are:

$$x = 5 + 4\lambda, \; y = -\lambda, \; z = 2 + 4\lambda$$

Rearrange each for λ:

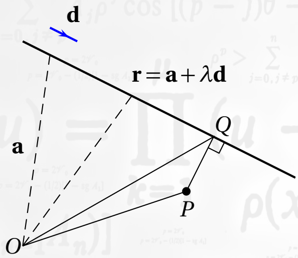
$$\lambda = \frac{x-5}{4}, \lambda = -y, \lambda = \frac{z-2}{4}$$

All three equations are equal to λ and so equal to one another:

$$\frac{x-5}{4} = -y = \frac{z-2}{4}$$

Shortest distance between a point and a line

Let's see how vector methods can solve the problem of finding the shortest distance of a given point (in 3D space) from a given line.



The picture here shows a line with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ and a point P.

The point Q is that point on the line closest to P.

We can calculate this point by knowing that the vector $\overrightarrow{PQ}$ is perpendicular to the line.

The direction of the line is given by $\mathbf{d}$ so the problem amounts to solving the scalar product equation

$$\mathbf{d}.\overrightarrow{PQ} = 0$$

Let's see how this works with an example.

Shortest distance from a point to a line

Find the shortest distance between the line

$$\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

and the point $P(1,1,2)$.

Referring to the picture on the previous slide, let Q be the point on the line closest to P. We need the vectors $\mathbf{d}$ and $\overrightarrow{PQ}$.

The direction vector is $\mathbf{d} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$ and we can find $\overrightarrow{PQ}$ from the picture on the previous slide:

$$\overrightarrow{PQ} = -\overrightarrow{OP} + \overrightarrow{OQ}$$

$\overrightarrow{OQ}$ is on the line, which means there is a value of λ corresponding to that point.

$$\begin{aligned}\overrightarrow{PQ} &= -\overrightarrow{OP}+\overrightarrow{OQ}\\ &= -\overrightarrow{OP}+\mathbf{a}+\lambda\mathbf{d}\\ \overrightarrow{PQ} &= -\begin{pmatrix}1\\1\\2\end{pmatrix}+\begin{pmatrix}4\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\5\end{pmatrix}=\begin{pmatrix}3+\lambda\\1+3\lambda\\2+5\lambda\end{pmatrix}\end{aligned}$$

The value of λ is that which satisfies $\mathbf{d}.\overrightarrow{PQ}=0$:

$$\begin{pmatrix}1\\3\\5\end{pmatrix}.\begin{pmatrix}3+\lambda\\1+3\lambda\\2+5\lambda\end{pmatrix}=0$$

Expanding the scalar product:

$$\begin{aligned}1(3+\lambda)+3(1+3\lambda)+5(2+5\lambda) &= 0\\ 35\lambda+16 &= 0 \qquad \therefore \lambda=-\frac{16}{35}\end{aligned}$$

Substituting λ into $\overrightarrow{PQ}$ gives $\overrightarrow{PQ} = \begin{pmatrix} 89/35 \\ -13/35 \\ -2/7 \end{pmatrix}$

We need the magnitude of this vector; it gives the smallest distance between P and Q.

$$PQ = \frac{3}{35}\sqrt{910} \approx 2.59\,\text{units}$$

The coordinates of Q are obtained by substituting λ into the line equation:

$$\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} - \frac{16}{35}\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 124/35 \\ 22/35 \\ 12/7 \end{pmatrix}$$

The point nearest the line is $Q(\frac{124}{35}, \frac{22}{35}, \frac{12}{7})$.

Test yourself

Try to answer the following questions.

1. Write the equation $y = 2x + 1$ in the form $\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$.
2. Given $\mathbf{r} = -2\mathbf{i} + \mathbf{j} + \lambda(3\mathbf{i} - \mathbf{j})$ find the parametric equations of the line.
3. Find the Cartesian equation of the line in (2).
4. Given that $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k})$ find the parametic equations of the line.
5. Find the symmetric equations of the line in (4).

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1. The line passes through $(0, 1)$ so one possibility is $\mathbf{r} = \mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j})$ but there are endless possible ways to do it!
2. $x = -2 + 3\lambda$, $y = 1 - \lambda$
3. $y = \frac{1}{3} - \frac{1}{3}x$ (or $x + 3y - 1 = 0$)
4. $x = 2 + 2\lambda$, $y = 2 - 5\lambda$, $z = -1 + 3\lambda$
5. $\frac{1}{2}(x - 2) = -\frac{1}{5}(y - 2) = \frac{1}{3}(z + 1)$