Scalar Product

Vectors 1

INU0114/514 (Maths 1)

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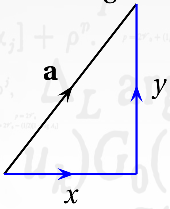
Objectives

In this presentation we'll look at an important concept in vector algebra: the scalar product.

- Scalar product of two vectors — the definition.
- Calculating the angle between two vectors
- Testing two vectors to check if they're perpendicular.

Recap: vector components

Vectors can be expressed in component form — in terms of perpendicular directions and lengths.



$$\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$$

This is called column vector notation.

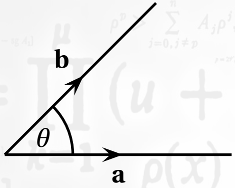
The same vector in Cartesian form is

$$\mathbf{a} = x\mathbf{i} + y\mathbf{j}$$

Scalar Product

The scalar product is a type of multiplication defined for two vectors. It's also called the *dot product*.

For n dimensional vectors **a** and **b** at angle θ:



$$\mathbf{a}.\mathbf{b} = ab\cos\theta = \sum_{i=1}^{n} a_i b_i$$

The scalar product is defined in two equivalent ways

- The first relates the magnitudes and angle between the vectors.
- The other shows how to calculate it by multiplying and adding the corresponding components of each vector.

The result is a single value (a scalar).

Scalar product (2D)

Two vectors are

$$\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} \text{ and } \mathbf{b} = -6\mathbf{i} + 2\mathbf{j}$$

Find the scalar product **a.b**

This is a simple case of multiplying corresponding components and adding:

$$\mathbf{a.b} = (3)(-6) + (-4)(2) = -18 - 8 = -26$$

Scalar product (3D)

Two vectors are given by

$$\mathbf{p} = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k} \text{ and } \mathbf{q} = 12\mathbf{j} - 4\mathbf{k}$$

Find the scalar product **p.q**

Multiply corresponding components and add:

$$\mathbf{p.q} = (-2)(0) + (1)(12) + (3)(-4) = 0 + 12 - 12 = 0$$

Angle between vectors

The scalar product allows calculation of the angle between two vectors.

$$\mathbf{a}.\mathbf{b} = ab\cos\theta$$

The scalar product formula is easily rearranged to make

$$\boxed{\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{ab}}$$

The denominator is the product of the vector magnitudes.

The numerator is calculated by multiplying and adding the vector components.

Angle between vectors

Two vectors are

$$\mathbf{a} = -5\mathbf{i} - 5\mathbf{j} \text{ and } \mathbf{b} = 2\mathbf{i} + 8\mathbf{j}$$

Find the angle between the vectors **a** and **b**.

First, we'll find the scalar product:

$$\mathbf{a}.\mathbf{b} = (-5)(2) + (-5)(8) = -10 - 40 = -50$$

We also need a and b — the magnitudes of the two vectors:

$$\begin{aligned} a &= \sqrt{(-5)^2 + (-5)^2} = \sqrt{50} \\ b &= \sqrt{2^2 + 8^2} = \sqrt{68} \end{aligned}$$

The angle is obtained from

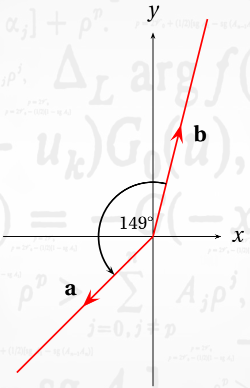
$$\cos\theta = \frac{-50}{\sqrt{50}\sqrt{68}} \approx -0.85749$$

Take the inverse cosine to get the angle:

$$\theta = 149.0^\circ \text{ (correct to 1 DP)}$$

Here is a picture of the vectors from this example.

$$\mathbf{a} = -5\mathbf{i} - 5\mathbf{j} \text{ and } \mathbf{b} = 2\mathbf{i} + 8\mathbf{j}$$



Compare this to the definition diagram for the scalar product seen earlier.

Angle between vectors (3D)

Two vectors are

$$\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k} \text{ and } \mathbf{b} = -\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$

Find the angle between the vectors.

First, we'll find the scalar product:

$$\mathbf{a}.\mathbf{b} = (3)(-1) + (1)(2) + (-2)(-2) = -3 + 2 + 4 = 3$$

We also need a and b — the magnitudes of the two vectors:

$$\begin{aligned} a &= \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14} \\ b &= \sqrt{(-1)^2 + 2^2 + (-2)^2} = 3 \end{aligned}$$

The angle is obtained from:

$$\cos\theta = \frac{3}{3\sqrt{14}} = \frac{1}{\sqrt{14}}$$

Take the inverse cosine to get the angle:

$$\theta = 74.5^\circ \text{ (correct to 1 DP)}$$

Perpendicular vectors

The scalar product for two vectors is

$$\mathbf{a}.\mathbf{b} = ab\cos\theta$$

When the vectors are perpendicular then $\theta = 90°$ and so

$$\mathbf{a}.\mathbf{b} = 0$$

This result can be a simple test to see if vectors are perpendicular or not.

Perpendicular?

Two vectors are

$$\mathbf{r} = 8\mathbf{i} + 12\mathbf{j} \text{ and } \mathbf{v} = -3\mathbf{i} + 2\mathbf{j}$$

Show that **r** and **v** are perpendicular vectors.

In this case

$$\mathbf{r}.\mathbf{v} = (8)(-3) + (12)(2) = -24 + 24 = 0$$

Therefore the vectors are perpendicular.

Test yourself

Now try the following questions.

Consider the vectors $\mathbf{a} = -4\mathbf{i} - 8\mathbf{j}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

Find the following:

1. $\mathbf{a}.\mathbf{b}$
2. $\mathbf{b}.\mathbf{c}$
3. The angle between **a** and **b**.
4. Given that $\mathbf{r} = 3m\mathbf{i} + 2\mathbf{j} + m\mathbf{k}$ is perpendicular to **c**, find the value of m.

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1. $\mathbf{a}.\mathbf{b} = -20$
2. $\mathbf{b}.\mathbf{c} = 4$
3. $\cos\theta = -1$, therefore $\theta = 180°$.
4. Perpendicular means $\mathbf{r}.\mathbf{c} = 0$.
   Therefore $6m + 2 - m = 0$ and so $m = -\frac{2}{5}$.