Vectors in 3D

Vectors 1

INU0114/514 (Maths 1)

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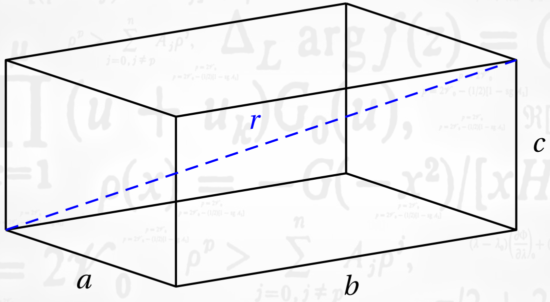
Objectives

This presentation will introduce some more vector concepts.

- Pythagoras's theorem in 3D
- Direction of a vector in 3D
- Direction cosines
- Unit vector in a given direction

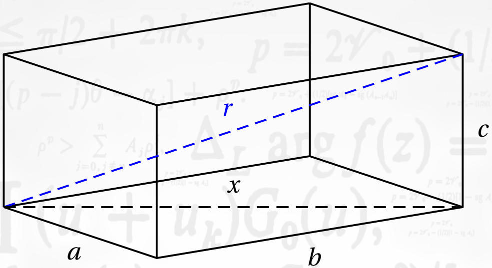
3D Geometry

The picture shows a cuboid; can you calculate the distance r?



The solution is found using Pythagoras's theorem....

We can make two right-angled triangles:



First: observe that $r^2 = x^2 + c^2$.

And also that $x^2 = a^2 + b^2$.

Combine these results to see that

$$\boxed{r^2 = a^2 + b^2 + c^2}$$

We'll need this result next!

Vectors in 3D

Vectors in 3D are easily represented in the notation seen so far. The unit base vectors are represented by **i**, **j** and **k** where

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

In Cartesian notation a 3D vector can be written as

$$\mathbf{p} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and the magnitude is:

$$p = \left|\mathbf{p}\right| = \sqrt{x^2 + y^2 + z^2}$$

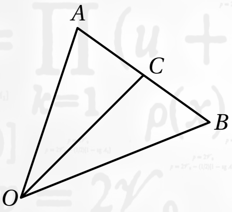
The direction of a 3D vector cannot be specified using a *single* angle.

We will see how to represent the direction of a 3D vector using a unit vector in the same direction later.

Position vector

The points A and B have coordinates $(1,4,-2)$ and $(-2,0,8)$ respectively. The point C is the midpoint of the line AB. Find the position vector of C in Cartesian form.

A picture often helps — it doesn't need to be drawn accurately.



The position vectors of A and B are

$$\overrightarrow{OA} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \text{ and } \overrightarrow{OB} = -2\mathbf{i} + 8\mathbf{k}$$

Also we see $\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$ so

$$\begin{aligned} \overrightarrow{AB} &= -(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) + (-2\mathbf{i} + 8\mathbf{k}) \\ &= -3\mathbf{i} - 4\mathbf{j} + 10\mathbf{k} \end{aligned}$$

The position vector of C is the vector $\overrightarrow{OC}$. In this case:

$$\begin{aligned} \overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AB} \\ &= \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \\ &\quad + \tfrac{1}{2}(-3\mathbf{i} - 4\mathbf{j} + 10\mathbf{k}) \\ \therefore \overrightarrow{OC} &= -\tfrac{1}{2}\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \end{aligned}$$

The coordinates are $C(-\frac{1}{2}, 2, 3)$.

Directions of 3D vectors

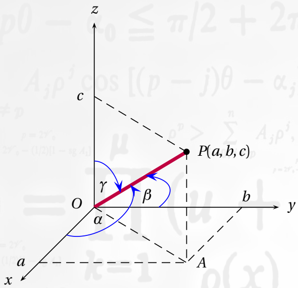
As already noted, the direction of a vector in three dimensions cannot be specified by a single angle.

There are two practical ways to specify directions of 3D vectors:

- Direction cosines
- Unit vectors in the same direction

We'll examine each of these over the next few slides.

Direction cosines



The picture shows a position vector $\mathbf{r} = \overrightarrow{OP}$.

The vector can be described by three angles (α, β and γ) with respect to the coordinate axes.

The angles and lengths involved are related through:

$$a = r\cos\alpha \quad \therefore \cos\alpha = \frac{a}{r}$$

$$b = r\cos\beta \quad \therefore \cos\beta = \frac{b}{r}$$

$$c = r\cos\gamma \quad \therefore \cos\gamma = \frac{c}{r}$$

Also the length of $\mathbf{r}$ is r and so

$$r^2 = a^2 + b^2 + c^2$$

These equations can be shown to give

$$\begin{aligned} r^2 &= (r\cos\alpha)^2 + (r\cos\beta)^2 + (r\cos\gamma)^2 \\ r^2 &= r^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) \\ 1 &= \cos^2\alpha + \cos^2\beta + \cos^2\gamma \end{aligned}$$

and these quantities are often abbreviated to

$$\boxed{l^2 + m^2 + n^2 = 1}$$

where $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.

The quantities l, m and n are referred to as the *direction cosines* of the vector.

They are the values of the cosines of the angles that the vector makes with the coordinate axes. The cosines of the angles are more often involved in calculations than the angles themselves.

For the vector

$$\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

We can find these directions quickly using

$$l = \frac{a}{r}, \quad m = \frac{b}{r}, \quad n = \frac{c}{r}$$

where $r = \sqrt{a^2 + b^2 + c^2}$.

Direction cosines

Find the direction cosines of the vector

$$\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

Here we have $a = 3$, $b = -2$ and $c = 6$ so that

$$r = \sqrt{3^2 + (-2)^2 + 6^2} = 7$$

Therefore the direction cosines are

$$l = \tfrac{3}{7}, \quad m = -\tfrac{2}{7}, \quad n = \tfrac{6}{7}$$

In terms of the earlier picture this means

$$\cos\alpha = \tfrac{3}{7}, \quad \cos\beta = -\tfrac{2}{7}, \quad \cos\gamma = \tfrac{6}{7}$$

Unit vector in same direction

We can also specify the direction of a vector using a unit vector.

We define the direction to be a unit vector called $\hat{\mathbf{r}}$ (spoken as spoken as r-hat!) where

$$\boxed{\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}} \tag{1}$$

This formula applies to any vector. Usually we don't use this definition with 2D vectors (because an angle is more useful). But for 3D vectors it's usually more helpful to give the direction using $\hat{\mathbf{r}}$.

Note that we can rearrange equation 1 to get

$$\mathbf{r} = r\hat{\mathbf{r}}$$

This means vectors can be thought of as being a magnitude multiplied by a direction.

Unit vector in same direction (2D)

Consider the vector $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j}$. Find a unit vector in the same direction.

The magnitude of $\mathbf{r}$ is given by:

$$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

When we multiply vector $\mathbf{r}$ by a factor of $\frac{1}{5}$ we will obtain the unit vector $\hat{\mathbf{r}}$. (Read or say that vector as "r-hat").

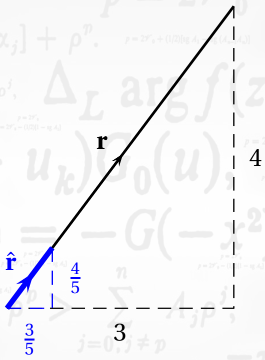
So

$$\hat{\mathbf{r}} = \frac{1}{5}\mathbf{r} = \frac{1}{5}(3\mathbf{i} + 4\mathbf{j}) \quad \therefore \hat{\mathbf{r}} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

The vector $\hat{\mathbf{r}}$ is the unit vector in the same direction as vector $\mathbf{r}$.

We wouldn't normally do this for a 2D vector; we can specify magnitude and angle just as easily.

This is how the vectors $\mathbf{r}$ and $\hat{\mathbf{r}}$ relate to one another from the previous example.



Unit vector $\hat{\mathbf{r}}$ points in the same direction as $\mathbf{r}$ but has length of 1 (which is exactly $\frac{1}{5}$ the length of $\mathbf{r}$).

For 3D vectors we must use either direction cosines or unit vectors to specify direction.

Unit vector in same direction (3D)

Consider the vector $\mathbf{p} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$. Find a unit vector in the same direction.

The magnitude of $\mathbf{p}$ is given by:

$$p = \sqrt{3^2 + (-2)^2 + 4^2} = \sqrt{9 + 4 + 16} = \sqrt{29}$$

When we multiply vector $\mathbf{p}$ by a factor of $\frac{1}{\sqrt{29}}$ we will obtain the unit vector $\hat{\mathbf{p}}$. (Read or say that vector as "p-hat").

So

$$\hat{\mathbf{p}} = \frac{1}{\sqrt{29}}\mathbf{p} = \frac{1}{\sqrt{29}}(3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = \frac{3}{\sqrt{29}}\mathbf{i} - \frac{2}{\sqrt{29}}\mathbf{j} + \frac{4}{\sqrt{29}}\mathbf{k}$$

The vector $\hat{\mathbf{p}}$ is the unit vector in the same direction as vector $\mathbf{p}$.

Test yourself

Now try to answer the following questions. Consider the vectors

$$\mathbf{v} = -\mathbf{i} + 4\mathbf{j} + 8\mathbf{k} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ \sqrt{3} \end{pmatrix}$$

1. Calculate $10\mathbf{v}$.
2. Calculate r.
3. Write down the vector $\hat{\mathbf{r}}$.
4. Find $\hat{\mathbf{v}}$ *and* give the direction cosines.

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1. $10\mathbf{v} = -10\mathbf{i} + 40\mathbf{j} + 80\mathbf{k}$.
2. $r = 4$
3. $\hat{\mathbf{r}} = \begin{pmatrix} 3/4 \\ -1/2 \\ \sqrt{3}/4 \end{pmatrix}$.
4. $\hat{\mathbf{v}} = -\frac{1}{9}\mathbf{i} + \frac{4}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}$.
   Direction cosines are $l = -\frac{1}{9}$, $m = \frac{4}{9}$, $n = \frac{8}{9}$.