Cartesian Vectors

Vectors 1

INU0114/514 (Maths 1)

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Objectives

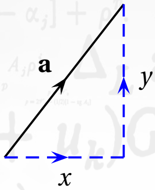
This presentation will introduce some more vector concepts.

- Cartesian and column vector notation for vector components
- Position vectors
- Magnitude of a vector
- Direction of a vector in 2D (an angle)

In the next presentation we will deal with 3D vectors...

Components of vectors

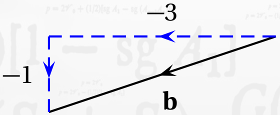
Vectors can be expressed in component form — in terms of perpendicular directions and distances.



$$\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$$

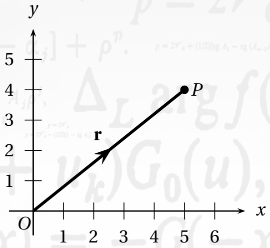
This is called column vector notation.

For example the vector $\mathbf{b} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ looks like this



Position vectors

Free vectors are vectors where starting position doesn't matter (e.g. on the previous slide). Only the size and direction are important.



When the starting position does matter the vector is called a position vector. The vector $\overrightarrow{OP}$, which is represented by a line segment with an arrow on it, is a position vector.

Position vectors are often denoted by **r**. In column vector notation the vector in the picture is given by

$$\mathbf{r} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

Standard basis vectors

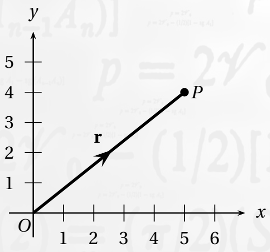
Definition

A unit vector is a vector with a magnitude (length) of 1 unit. A standard base vector is a vector with magnitude of 1 unit in the same direction as an axis.

In two dimensions the unit base vectors are represented by **i** and **j** where

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

And so **i** is a step of one unit along the positive direction of the x-axis and **j** is one unit step in the positive direction of the y-axis by **j**.



The position vector shown here is:

$$\mathbf{r} = 5\mathbf{i} + 4\mathbf{j}$$

In general, a point (x, y) can be specified with a position vector of the form:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} \quad \text{or } \mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Vector arithmetic

Addition, subtraction and multiplying by a scalar

Given the vectors:

$$\begin{aligned}\mathbf{a} &= 2\mathbf{i} - 4\mathbf{j} \\ \mathbf{b} &= -3\mathbf{i} + 7\mathbf{j}\end{aligned}$$

Find (a) $\mathbf{a}+\mathbf{b}$ (b) $\mathbf{b}-\mathbf{a}$ (c) $-5\mathbf{b}$

(a) Add the corresponding components

$$\begin{aligned}\mathbf{a}+\mathbf{b} &= (2-3)\mathbf{i} + (-4+7)\mathbf{j} \\ &= -\mathbf{i} + 3\mathbf{j}\end{aligned}$$

(b) Subtract the corresponding components

$$\begin{aligned}\mathbf{b}-\mathbf{a} &= (-3-2)\mathbf{i} + (7-(-4))\mathbf{j} \\ &= -5\mathbf{i} + 11\mathbf{j}\end{aligned}$$

(c) Multiply both components by -5:

$$\begin{aligned}-5\mathbf{b} &= -5(-3\mathbf{i} + 7\mathbf{j}) \\ &= 15\mathbf{i} - 35\mathbf{j}\end{aligned}$$

Arithmetic with column vectors

Given the vectors:

$$\mathbf{a} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

Find $3\mathbf{a} - 2\mathbf{b}$.

In column vector notation we can write

$$\begin{aligned} 3\mathbf{a} - 2\mathbf{b} &= 3\begin{pmatrix} 2 \\ -3 \end{pmatrix} - 2\begin{pmatrix} 5 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -9 \end{pmatrix} + \begin{pmatrix} -10 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ -13 \end{pmatrix} \end{aligned}$$

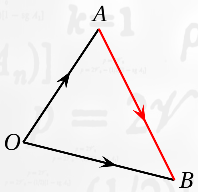
Position vectors

The points $A(10,5)$ and $B(-4,1)$ are in a 2D plane.
(a) Write down the position vectors of $\overrightarrow{OA}$ and $\overrightarrow{OB}$
(b) Find the vector $\overrightarrow{AB}$.

(a) In column vector notation we have

$$\overrightarrow{OA} = \begin{pmatrix} 10 \\ 5 \end{pmatrix} \text{ and } \overrightarrow{OB} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

(b) A picture of these vectors looks like this:



(This is not drawn accurately).

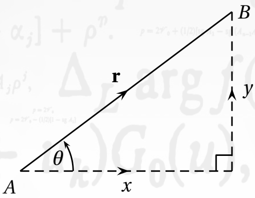
The picture shows how the vectors are related. Recall that $\overrightarrow{OA}$ means 'go from O to A'.

The picture shows that $\overrightarrow{OA} = -\overrightarrow{AB} + \overrightarrow{OB}$.

$$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 10 \\ 5 \end{pmatrix} = \begin{pmatrix} -14 \\ -4 \end{pmatrix}$$

Magnitude of a vector

The diagram below shows the vector **r** from point A to point B.



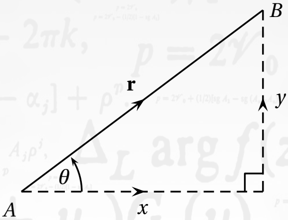
The vector, in Cartesian form, is $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

The magnitude of the vector is the same as the length of the line, **r**.

Using Pythagoras' Theorem the magnitude is given by

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2}$$

Direction of a vector



The direction of the vector **r** is the angle θ measured in an anticlockwise direction from the positive x-axis.

This can be calculated from:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Care must be taken to ensure the angle θ is given in the correct quadrant.

We will follow a convention for expressing the angle of a vector: it will be expressed as $0° \leq \theta < 360°$ or $0 \leq \theta < 2\pi$.

Recap: solving tangent equations

Recall from earlier in the course that the equation

$$\tan\theta = k$$

is solved by finding the principal value $PV = \tan^{-1} k$.

Then the general solution is given by

$$\theta = 180n + PV$$

where n is an integer.

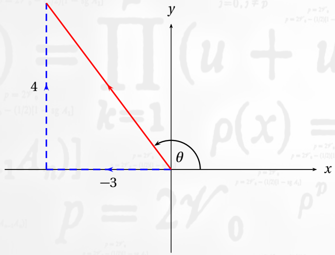
In other words – we find $\tan^{-1} k$ and then add 0, 180, 360, etc. to get the other solutions.

We must remember this behaviour of the general solution when finding the angles of vectors!

Getting the correct direction...

Find the direction of $\mathbf{a} = -3\mathbf{i} + 4\mathbf{j}$.

First — make sketch the of the vector.



The vector lies in the 2nd quadrant (between 90°and 180°).

To find the angle:

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{y}{x}\right) \\ &= \tan^{-1}\left(\frac{4}{-3}\right) \\ &= -53.1^\circ \end{aligned}$$

But this is in the 4th quadrant.

We must add 180° to get to the 2nd quadrant:

$$\theta = -53.1^\circ + 180^\circ = 126.9^\circ$$

A particle is moving with a velocity vector given by

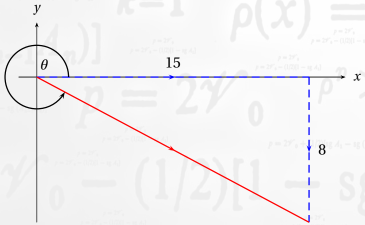
$$\mathbf{v} = (15\mathbf{i} - 8\mathbf{j})\,\mathrm{m\,s^{-1}}$$

Calculate the speed and direction (measured in degrees from the positive x-axis).

The speed of the particle is found from Pythagoras' Theorem:

$$v = |\mathbf{v}| = \sqrt{15^2 + (-8)^2} = \sqrt{289} = 17\,\mathrm{m\ s^{-1}}$$

We can find the angle from a sketch of the vector:



The angle θ, in the fourth quadrant.

$$\theta = \tan^{-1}\left(\frac{-8}{15}\right) = -28.1^\circ$$

Add 360° to get the angle θ into the correct interval $(0 \le \theta < 360^\circ)$.

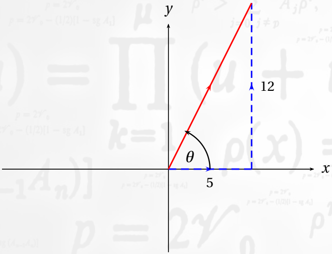
$$\theta = -28.1^\circ + 360^\circ = 331.9^\circ$$

The speed of the particle is $17\,\mathrm{m\,s^{-1}}$ at an angle of $\theta = 331.9^\circ$.

Find the magnitude and direction of the force $\mathbf{F} = (5\mathbf{i} + 12\mathbf{j})$ N.

The magnitude is $F = \sqrt{5^2 + 12^2} = 13$ N.

The sketch of this vector:



The vector lies in the 1st quadrant (between 90°and 180°).

To find the angle:

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{y}{x}\right) \\ &= \tan^{-1}\left(\frac{12}{5}\right) \\ &= 67.4^\circ \end{aligned}$$

This is in the 1st quadrant – no need to change anything!

Test yourself

Now try to answer the following questions. Consider the vectors $\mathbf{p} = 3\mathbf{i} - 8\mathbf{j}$, $\mathbf{q} = -8\mathbf{i} + 8\mathbf{j}$ and $\mathbf{r} = 12\mathbf{i} + 5\mathbf{j}$

Find the following:

1. $\mathbf{p} + \mathbf{q} + \mathbf{r}$
2. $3\mathbf{p}$
3. Magnitude and direction of $\mathbf{r}$
4. Magnitude and direction of $\mathbf{q}$

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1. $\mathbf{p} + \mathbf{q} + \mathbf{r} = 7\mathbf{i} + 5\mathbf{j}$
2. $3\mathbf{p} = 9\mathbf{i} - 24\mathbf{j}$
3. $r = 13,\ \theta = 22.6°$
4. $q = 8\sqrt{2},\ \theta = 135°$