Introduction to Vectors

Vectors 1

INU0114/514 (Maths 1)

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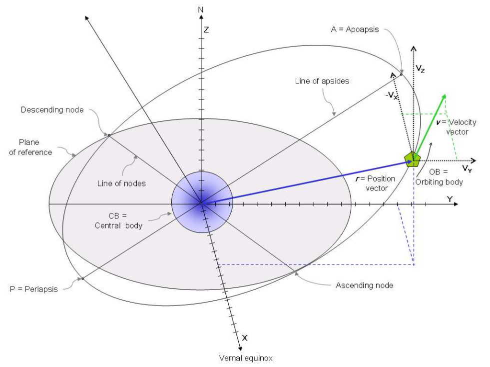
Introduction



Vectors are one of the most powerful tools in mathematics.

They can represent quantities such as displacement, velocity and forces using directed line segments.

Representing quantities in this way allows us to analyse them using trigonometry and geometry.

Objectives

This presentation will introduce some basic concepts in vector algebra.

- Vectors and scalars
- Geometric representation of a vector
- Parallel vectors
- Equality of vectors
- Triangle and parallelogram rules
- Resultant vectors

In Maths 2 you will also learn to differentiate and integrate vectors!

Vector and scalar quantities

Scalar quantities are completely described by a single number representing the magnitude (the size) of the quantity. They do not have a direction.

Examples of scalar quantities are time, mass, electric charge, speed, distance and temperature.

Vector quantities must be described by a magnitude (length or size) and a direction.

Examples of vector quantities are force, velocity, displacement, momentum, thrust and acceleration.

Representation

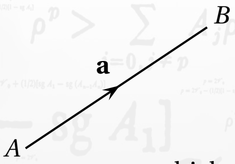
Vectors are represented in print using a bold font; for example the vector $\mathbf{a}$.

In handwritten mathematics, it is usual to indicate vectors by an underline. For example the vector $\underline{a}$.

The magnitude of the vector $\mathbf{a}$ is represented by a or $|\mathbf{a}|$.

A vector can be drawn as an arrow.

- The length of the arrow can represent the magnitude.
- The arrow itself indicates the direction.



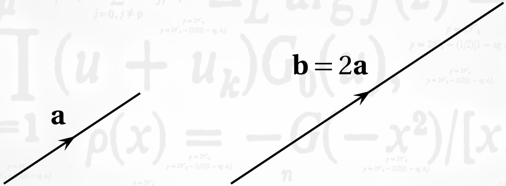
The picture above represents a vector, which can be denoted by $\overrightarrow{AB}$ or by $\mathbf{a}$.

Parallel vectors

Two vectors **a** and **b** are parallel if one is a scalar multiple of the other, i.e., if

$$\mathbf{b} = \lambda \mathbf{a}$$

If λ is positive then the vectors are in the same direction.



If λ is negative then the vectors are in opposite directions.

Equal vectors

Two vectors are said to be equal if they have the same magnitude and direction.

The picture shows parallel vectors, such that

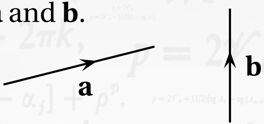
$$\mathbf{a} = \mathbf{b}$$

Since the vectors are equal, then the magnitudes must also be equal:

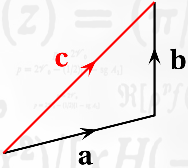
$$a = b$$

Vector addition: Triangle Law

Consider the vectors **a** and **b**.



Add the vectors by placing the start of vector **b** at the end of vector **a**.



A new vector **c** represents the equivalent vector journey from the start of **a** to the end of **b**.

In vector terms we can write

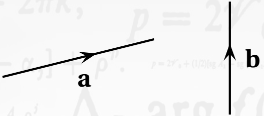
$$\mathbf{c} = \mathbf{a} + \mathbf{b}$$

This is called the triangle law of addition.

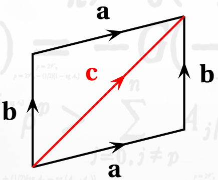
The vector **c** is the resultant vector obtained from adding **a** and **b**.

Vector addition: Parallelogram Law

Consider the vectors **a** and **b**.



The order of addition doesn't matter: following vector **b** first with vector **a** will produce the same result.



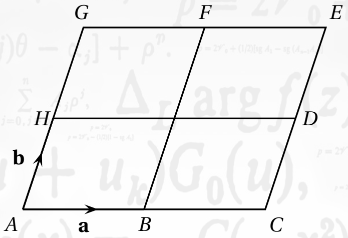
This is the parallelogram law of addition. It states

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}=\mathbf{c}$$

The vector **c** is the resultant vector obtained from adding **a** and **b**.

Resultant vectors

The picture below shows a *parallelogram ACEG*



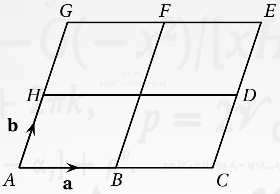
The points *B, D, F* and *H* are the midpoints of each side.

Two vectors **a** and **b** are defined so that

$$\mathbf{a} = \overrightarrow{AB} \qquad \text{and}\, \mathbf{b} = \overrightarrow{AH}$$

We can express all other vectors in this shape in terms of **a** and **b**.

We just need to look for parallel sides...



Let's express some of these vectors in terms of **a** and **b**.

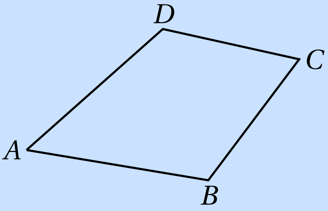
- $\overrightarrow{AC} = \mathbf{a} + \mathbf{a} = 2\mathbf{a}$.
- $\overrightarrow{BH} = -\mathbf{a} + \mathbf{b}$ (or $\mathbf{b} - \mathbf{a}$)
- $\overrightarrow{AF} = \mathbf{a} + \mathbf{b} + \mathbf{b} = \mathbf{a} + 2\mathbf{b}$.
- $\overrightarrow{EH} = -\mathbf{a} - \mathbf{a} - \mathbf{b} = -2\mathbf{a} - \mathbf{b}$.

Notice that it doesn't matter how you get from the start to the end of the journey:

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AH} + \overrightarrow{GH} + \overrightarrow{GF} + \overrightarrow{FE} + \overrightarrow{DC} \\ &= \mathbf{b} + \mathbf{b} + \mathbf{a} + \mathbf{a} - \mathbf{b} - \mathbf{b} \\ &= 2\mathbf{a}\end{aligned}$$

Resultant vectors

Consider the quadralateral $ABCD$



Write down a single resultant vector to replace the following:

(a) $\overrightarrow{AB}+\overrightarrow{BC}$ (b) $\overrightarrow{BC}+\overrightarrow{CD}$ (c) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$ (d) $\overrightarrow{AB}+\overrightarrow{DA}$

(a) $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

(b) $\overrightarrow{BC}+\overrightarrow{CD}=\overrightarrow{BD}$

(c) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}=\overrightarrow{AD}$ (or $-\overrightarrow{DA}$).

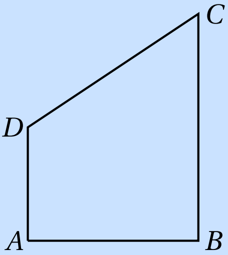
(d) Change the order of addition and it's easy!

$\overrightarrow{AB}+\overrightarrow{DA}$ is the same as $\overrightarrow{DA}+\overrightarrow{AB}$.

In the picture this obviously the same as $\overrightarrow{DB}$ (or $-\overrightarrow{BD}$).

Vector addition

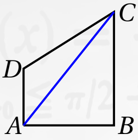
The picture shows a trapezium $ABCD$



where $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{BC}$ and $\overrightarrow{AD} = \frac{1}{2}\mathbf{b}$.

Find, in terms of **a** and **b** the following vectors:

(a) $\overrightarrow{AC}$ (b) $\overrightarrow{CD}$ (c) $\overrightarrow{BD}$.

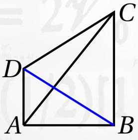


(a) It's obvious from the picture that

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{BC} \\ &= \mathbf{a} + \mathbf{b}\end{aligned}$$

(b) Again, with reference to the picture above:

$$\begin{aligned}\overrightarrow{CD} &= \overrightarrow{CA} + \overrightarrow{AD} \\ &= -\overrightarrow{AC} + \overrightarrow{AD} \\ &= -(\mathbf{a} + \mathbf{b}) + \tfrac{1}{2}\mathbf{b} \\ &= -\mathbf{a} - \tfrac{1}{2}\mathbf{b}\end{aligned}$$

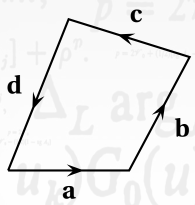


(c) The required vector is shown:

$$\begin{aligned}\overrightarrow{BD} &= \overrightarrow{BA} + \overrightarrow{AD} \\ &= -\overrightarrow{AB} + \overrightarrow{AD} \\ &= -\mathbf{a} + \tfrac{1}{2}\mathbf{b}\end{aligned}$$

The Zero Vector

The following picture shows the addition of four vectors.



Adding all four vectors returns to the start of the first vector.

$$\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = \mathbf{0}$$

The quantity on the RHS is a vector - called the zero vector.

The magnitude of the zero vector is 0.

The zero vector is written as $\underline{0}$ when handwriting it.

The zero vector is a useful concept if we need to consider how to balance a number of forces (so that they're in equilibrium).