SINE AND COSINE RULES

TRIGONOMETRY 2

INU0115/515 (MATHS 2)

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Objectives

In this presentation we're going to learn some methods to solve triangle problems. That means when we're given information about a triangle, such as the length of some sides or some angles, we can calculate any missing sides and angles.

You will probably have done this in the past with right angle triangles; they can be solved fairly easily using Pythagoras's theorem.

When the triangle *does not* contain a right angle we might be able to use

- Sine rule, or
- Cosine rule

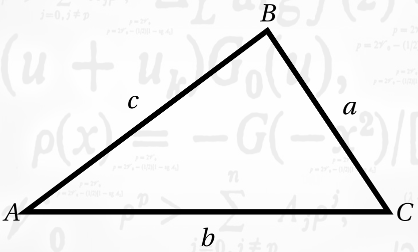
instead. These rules are easy to apply — the difficulty might be deciding which rule to use. Also, if the triangle information is incomplete we might be able to calculate more than one possible triangle; this is the ambiguous case.

Later in the presentation we'll also see a useful formula for calculating triangle areas.

Labelling convention

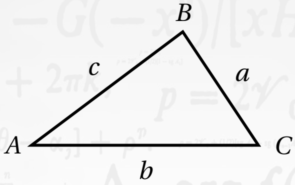
Consider the general triangle ABC shown in the picture.

- The angles are labelled with uppercase letters.
- The sides opposite have corresponding lowercase letters.



We'll follow this labelling convention in our discussions of triangles, particularly in regard to the Sine Rule and Cosine Rule.

The Sine Rule



The sine rule relates opposite sides and angles by the formula:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

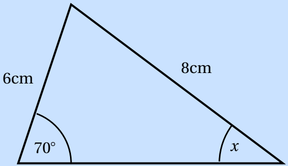
Or equivalently

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The sine rule can be used to solve any triangle provided we know

- Two angles and one side , or
- Two sides and an angle that is not included between the sides.

The sine rule (two sides and a non-included angle)



Calculate the angle x to the nearest 0.1°.

Write out the sine rule with the known sides and angles:

$$\begin{aligned} \frac{\sin x}{6} &= \frac{\sin 70^\circ}{8} \\ \sin x &= \frac{6 \sin 70^\circ}{8} \\ &= 0.704769 \end{aligned}$$

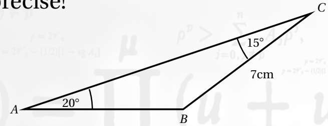
The principal value of x, using a calculator, is:

$$x = \sin^{-1}(0.704769) = 44.8^\circ$$

The sine rule (1 side and 2 angles)

A triangle has sidelength $BC = 7$cm and angles $A = 20°$ and $C = 15°$. Calculate the lengths of AB and AC to one decimal place.

In problems about triangles it is a good idea to sketch the triangle. It needn't be precise!



C is found by subtracting the other two angles from 180°.

$$C = 180 - 20 - 15 = 145°$$

The length of AB is found from:

$$\frac{AB}{\sin 15} = \frac{7}{\sin 20}$$

Rearrange to get

$$AB = \frac{7\sin 15}{\sin 20} = \frac{7 \times 0.2588}{0.3420} = 5.297$$

So $AB = 5.3$cm, to 1D.P.

Use the sine rule again to find AC:

$$\frac{AC}{\sin 145} = \frac{7}{\sin 20}$$

Rearrange

$$AC = \frac{7\sin 145}{\sin 20} = \frac{7 \times 0.5736}{0.3420} = 11.7392$$

So $AC = 11.7$cm, to 1D.P.

The ambiguous case

Sometimes, when two sides and one angle are given then two different triangles can be found from the given information. This situation is called the *ambiguous case*.

In the previous example the given triangle had two angles specified (15° and 20°). This meant that remaining angle must have been $180 - 15 - 20 = 145°$. There was no ambiguity in that case.

The ambiguous case cannot occur if two of the three angles are given.

The ambiguous case also *cannot* occur if two sides and the *included* angle are given. Can you understand why?

In problems where two sides and the *non-included* angle is given then we should check for the ambiguous case.

We'll examine some triangles fitting this description next.

Testing for the ambiguous case

Given the triangle in which $A = 25°$, $a = 12$cm and $c = 27$cm. Is this information enough to specify a unique triangle?

Since we know a, A and c we can find C using:

$$\frac{\sin C}{c} = \frac{\sin A}{a} \qquad \Rightarrow \sin C = \frac{27 \sin 25}{12} \qquad \therefore \sin C = 0.950891$$

The principal value of C

$$C = \sin^{-1}(0.950891) = 72°$$

But there is a secondary value *which has the same sine as this angle*

$$C = 180 - 72° = 108°$$

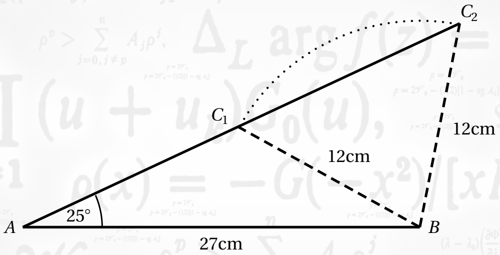
We should check both C angles to see if they are consistent with the given dimensions of the triangle.

1. If $C = 72°$ and given that $A = 25°$ then $B = 83°$.
2. If $C = 108°$ and given that $A = 25°$ then $B = 47°$.

Since two triangles are possible using the given information, then the information was ambiguous.

Recall that we were given $A = 25°$, $a = 12$ cm and $c = 27$ cm.

The picture below shows why the triangle given in the previous example is ambiguous.



The dashed line shows *two possible positions* that the side a can be whilst still being consistent with given triangle information.

An ambiguous case?

Given the triangle in which $A = 37°$, $a = 4.6$cm and $c = 2.1$cm. Is this information enough to specify a unique triangle?

The angle C is found from

$$\frac{\sin C}{2.1} = \frac{\sin 37}{4.6} \quad \Rightarrow \sin C = \frac{2.1 \sin 37}{4.6} = 0.2747$$

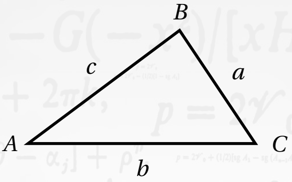
The principle value for this is $C = \sin^{-1}(0.2747) = 15.9°$.

The secondary value should be checked and it is $C = 180 - 15.9 = 164.1°$.

1. If $C = 15.9°$ and we know $A = 37°$ then we calculate $B = 127.1°$.
2. If $C = 164.1°$ and we know $A = 37°$ then we can't calculate B because $A + C > 180°$.

This time there was only one triangle consistent with the information given. The information was not ambiguous.

The Cosine Rule



The sides and opposite angles of a triangle are related through the cosine rule. The cosine rule is:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Or we could have chosen to calculate b or c using equivalent formulas:.

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

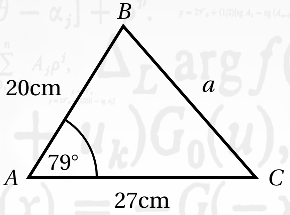
We can use the cosine rule to find missing sides and/or angles in the following situations:

- All three sides are known
- Two sides and the angle between them are given.

Cosine rule (two sides and the included angle)

Given the triangle with $A = 79°$, $b = 27$ cm and $c = 20$ cm, calculate the length of the missing side a.

Begin with the sketch.



The cosine rule $a^2 = b^2 + c^2 - 2bc \cos A$

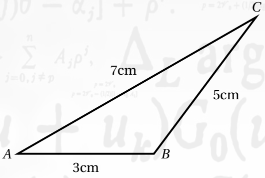
$$\begin{aligned} \therefore a^2 &= 27^2 + 20^2 - 2(27)(20)\cos 79° \\ &= 1129 - 206.07372 \\ a^2 &= 922.9263 \end{aligned}$$

Take the square-root to get $a = 30.4$cm (to 1DP).

Cosine rule (three sides given)

A triangle has side lengths of $a = 5$cm, $b = 7$cm and $c = 3$cm. Find the *largest angle* in the triangle.

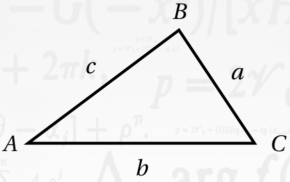
As usual, make a sketch — it helps to see what's going on.



The largest angle is opposite the longest side - it's angle B. The cosine rule formula gives

$$\begin{aligned} 7^2 &= 3^2 + 5^2 - 2(3)(5)\cos B \\ 49 &= 34 - 30\cos B \\ 15 &= -30\cos B \\ -\tfrac{1}{2} &= \cos B \quad \therefore B = \cos^{-1}(-\tfrac{1}{2}) = 120^\circ \end{aligned}$$

Area of a triangle



There are many, many formulae which give the area of a triangle. The following will be useful on this course:

$$\text{Area} = \begin{cases} \frac{1}{2} bc \sin A \\ \frac{1}{2} ac \sin B \\ \frac{1}{2} ab \sin C \end{cases}$$

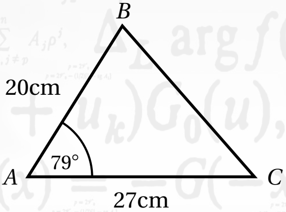
It isn't necessary to memorise these three formulae. These relationships imply that, if we know two sides of the triangle and the angle between them, then we can calculate the area of the triangle. So an alternative way of remembering these formulae could be

$$\text{Area} = \frac{1}{2} \times (\text{Side 1}) \times (\text{Side 2}) \times \sin(\text{Included Angle})$$

Using the area formula

Calculate the area of the triangle specified with $A = 79°$, $b = 27$cm and $c = 20$cm. Give the area to the nearest cm^2.

This triangle was seen in an earlier example. Here is the sketch.



$$\begin{aligned} \text{Area} &= \tfrac{1}{2} \times 20 \times 27 \times \sin 79° \\ &= 270 \sin 79° \\ &= 270 \times 0.9816 \\ &= 265.0393 \end{aligned}$$

So the area of this triangle, to the nearest cm^2, is 265cm^2.

Summary

In the solution of triangles the information about sides and angles given at the start decides which rule should be used:

- Two angles and one side (Sine rule)
- Two sides and an opposite angle (Sine rule)
- Three sides (Cosine rule)
- Two sides and the included angle (Cosine rule)

Note also that you can use both methods on many triangle problems; e.g. after using the cosine rule to find a missing angle, you can switch to sine rule to find the other sides and angles if you wish.

Test yourself...

Choose an appropriate method to solve the following triangles. Give answers to 1 DP if an exact value can't be found.

1. $A = 45°$, $b = 10$ cm, $c = 20$ cm. Find a.
2. $a = 9$ cm, $b = 12$ cm, $B = 30°$. Find A.
3. Is the triangle in (2) ambiguous?
4. $a = 8$ cm, $c = 11$ cm, $A = 25°$. Is this ambiguous?
5. Calculate the area of the triangle in (1).
6. $a = 14$ cm, $b = 8.5$ cm, $c = 4.5$ cm. Find B.

Answers:

1. Cosine rule. $a = 14.7$ cm
2. Sine rule. $A = 22.0°$
3. No.
4. Yes; two possible triangles.
5. Area $= 50\sqrt{2}$ cm^2 (≈ 70.7 cm^2)
6. No solution; not a triangle :-p