Trig equations (2)

Trigonometry 1

INU0115/515 (Maths 2)

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Reciprocal trig functions

We have seen three fundamental trig functions: $\sin\theta$, $\cos\theta$ and $\tan\theta$.

There are three related trigonometric ratios that you need to know.

They are called cosecant, secant and cotangent and they are defined as follows.

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \tag{1}$$

$$\sec\theta = \frac{1}{\cos\theta} \tag{2}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} \tag{3}$$

Note that the cosecant function is sometimes shortened to $\csc\theta$ in some mathematical texts. The reciprocal ratios usually don't appear on scientific calculators but they are easy to calculate.

Evaluating reciprocal trig functions

Cosecant

Use your calculator to find $\operatorname{cosec} 30^\circ$.

$$\operatorname{cosec} 30^\circ = \frac{1}{\sin 30^\circ} = \frac{1}{0.5} = 2$$

Secant

Evaluate $\sec 225^\circ$.

$$\sec 225^\circ = \frac{1}{\cos 225^\circ} = \frac{1}{-\sqrt{2}/2} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

Cotangent

Calculate $\cot 90^\circ$.

We can't use $\cot 90^\circ = \dfrac{1}{\tan 90^\circ}$ because $\tan 90^\circ$ is undefined.

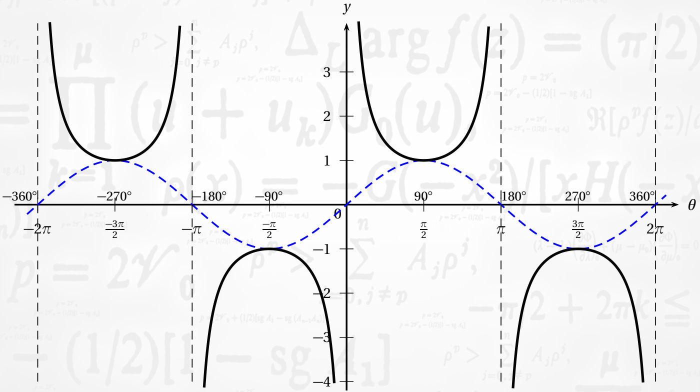
However, in this case we use the other definition:

$$\cot 90^\circ = \frac{\cos 90^\circ}{\sin 90^\circ} = \frac{0}{1} = 0$$

Graphs of reciprocal trig functions

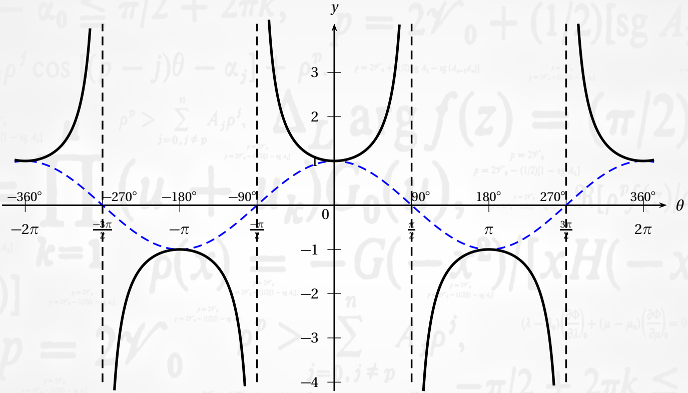
The next few slides will compare the graphs of reciprocal trig functions to sine, cosine and tangent functions.

Here's the graph of $y = \operatorname{cosec}\theta$ (compared to $y = \sin\theta$)



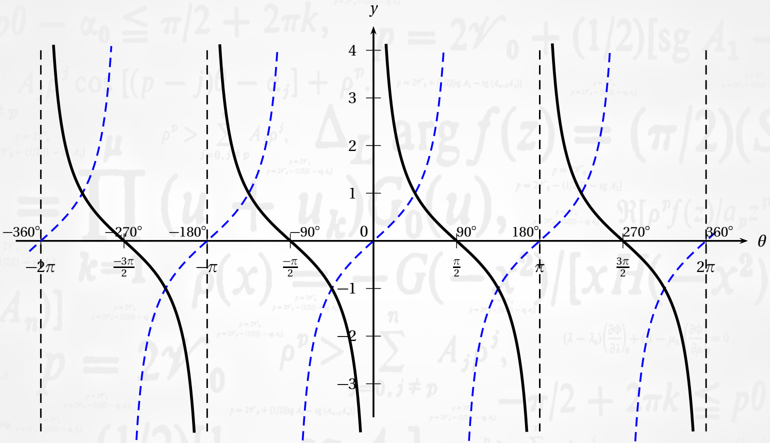
Notice how the graph has asymptotes at the places where $\sin\theta = 0$.

Here's the graph of $y = \sec\theta$ (compared to $y = \cos\theta$).



Notice how the graph has asymptotes at the places where $\cos\theta = 0$.

Here's the graph of $y = \cot\theta$ (compared to $y = \tan\theta$).



The graph has asymptotes at the places where $\tan\theta = 0$ (and $\sin\theta = 0$).

Equations with reciprocal trig functions

You should already know how to solve simple equations involving sine, cosine and tangent.

Solving a simple equation with secant

Solve $\sec\theta = 2$, $0 < \theta < 720°$.

We can express $\sec\theta$ using $\cos\theta$ and rewrite the equation as:

$$\frac{1}{\cos\theta} = 2$$

Rearrange:

$$\therefore \cos\theta = \tfrac{1}{2}$$

We solve this in the usual way.
First, find the principal value

$$PV = \cos^{-1}\left(\tfrac{1}{2}\right) = 60°$$

Now write down the general solution for cosine:

$$\theta = 360° n \pm 60°$$

Next , we substitute values of n until we have all the required solutions
$n = 0$ gives $\theta = \pm 60°$.
$n = 1$ gives $\theta = 300°$ and $420°$.
$n = 2$ gives $\theta = 660°$ and $780°$.
The set of solutions to our original equation consists of

$$\theta = 60°, 300°, 420° \text{ and } 660°$$

Multiple angle functions

So far, we have solved trig equations where the argument has been θ (or x, etc). We will spend some time looking at cases where the argument is more complicated than this!

Equation with a multiple angle

Solve the equation $\tan 3\theta = -\sqrt{3}$ in $0 < \theta < 180°$.

We begin in the same way as before - with the principal value.

$$PV = \tan^{-1}(-\sqrt{3}) = -60°$$

Now, we need the general solution for the tangent function: this is where the solution differs from earlier examples.

$$\begin{aligned} 3\theta &= 180n + PV \\ &= 180n - 60° \end{aligned}$$

We are looking for the solution θ so divide both sides by 3:

$$\theta = 60n - 20°$$

Substituting values of $n = 1, 2, 3$ gives three solutions in the interval:

$$\theta = 40°,\ 100°,\ 160°$$

Equation with a multiple angle

Solve the equation $\sin(2\theta + 40°) = \frac{1}{2}$, $-180° < \theta < 180°$.

We'll find the principal and secondary values for the RHS:

$$PV = \sin^{-1}(\tfrac{1}{2}) = 30° \quad \text{and } SV = 180° - PV = 150°$$

The general solution formulas for sine give:

$$2\theta + 40 = 360n + 30 \quad \text{and } 2\theta + 40 = 360n + 150$$

Rearrange each equation to make θ the subject:

$$\theta = 180n - 5 \quad \text{and } \theta = 180n + 55$$

Substitute values of n until we have all the angles in the given interval

$$\therefore \theta = -125°, -5°, 55°, 175°$$

A quadratic trig equation

Solve $2\cos^2\theta - 7\cos\theta - 4 = 0$, $0 < \theta < 360°$.

This equation is a quadratic in $\cos\theta$

$$\left[\text{Quadratic: } 2c^2 - 7c - 4 = 0 \quad \text{where } c \equiv \cos\theta\right]$$

This quadratic can be factorised $(2c+1)(c-4) = 0$; that means the original equation can be too:

$$(2\cos\theta + 1)(\cos\theta - 4) = 0$$

Therefore we have two equations to consider:

$$2\cos\theta + 1 = 0 \Rightarrow \cos\theta = -\tfrac{1}{2}$$

This is solved in the usual way to give $\theta = 120°$ and $\theta = 240°$.

The other possibility is

$$\cos\theta - 4 = 0 \Rightarrow \cos\theta = 4$$

However there are no solutions to this; because $-1 \le \cos\theta \le 1$.

Therefore

$$\therefore \theta = 120°, 240°$$

Test yourself

Solve the following equations and give all solutions in the specified interval.

1. $\cot x = \sqrt{3}$, $0 < x < 360°$.
2. $\operatorname{cosec} x = \sqrt{2}$, $-360° < x < 360°$
3. $\cos 3x = \frac{1}{2}$, $0 \le x \le 180°$
4. $\sin(x - 50°) = 1$, $0 \le x \le 360°$
5. $\sec(2x + \frac{1}{4}\pi) = -2$, $0 < x < \pi$.
6. $\tan^2 x - \tan x - 2 = 0$, $0 < x < 360°$. (Answers to 1 DP.)

Answers:

1. $x = 30°$ and $210°$.
2. $x = -315°, -225°, 45°$ and $135°$.
3. $x = 20°, 100°$ and $140°$.
4. $x = 140°$.
5. $x = \frac{11\pi}{24}$ and $\frac{19\pi}{24}$.
6. $x = 63.4°, 135°, 243.4°$ and $315°$.