Trig equations (1)

Trigonometry 1

INU0115/515 (Maths 2)

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Solutions of trigonometric equations

In this presentation we'll learn how to solve equations of the form

$$\sin\theta = k \qquad \cos\theta = k \qquad \tan\theta = k$$

where k is a constant.

The graphs of trig functions are periodic (sine and cosine repeat with a period of 360° and tangent with a period of 180°) so there are an infinite number of solutions to any trigonometric equation.

Usually when we solve a trig equation we restrict solution (the value of θ) to specific range, e.g. $0 \leq x < 360°$.

Inverse trigonometric functions

Let's understand what it means to solve a trig equation.

A function maps an input value to an output value using a rule. For example the sine function takes an input angle and returns an output value:

$$30^\circ \longrightarrow \boxed{\sin 30^\circ} \longmapsto \frac{1}{2}$$

"sine function"

Inverse functions 'undo' the effects of the original function.

$$\frac{1}{2} \longrightarrow \boxed{\sin^{-1} 30^\circ} \longmapsto 30^\circ$$

"inverse sine function"

Inverse functions for sine, cosine and tangent are programmed into scientific calculators.

- Inverse sine: $\sin^{-1} k$ (sometimes written $\arcsin k$)
- Inverse cosine: $\cos^{-1} k$ (sometimes written $\arccos k$)
- Inverse tangent: $\tan^{-1} k$ (sometimes written $\arctan k$)

The -1 here is *not* a power; it is notation for an inverse function.

Inverse sine

Use your calculator to find $\sin^{-1}(\frac{5}{6})$.

Easy! $\sin^{-1}(\frac{5}{6}) = 56.4°$ (to one decimal place).

Inverse cosine

Given that $\cos\theta = 0.4$, use your calculator to find a value for θ.

In this case we have $\theta = \cos^{-1}(0.4) = 66.4°$ to one decimal place.

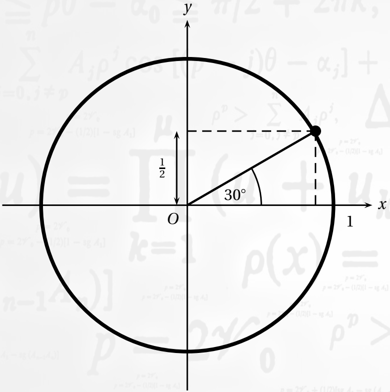
Inverse tangent

Given that $\tan\theta = -\sqrt{3}$, use your calculator to find a value for θ.

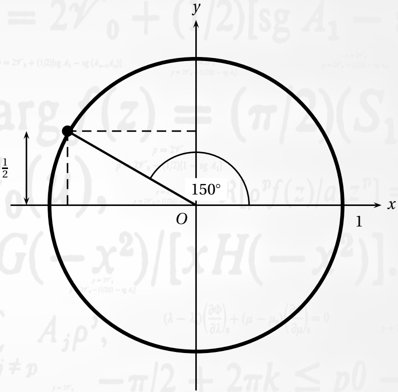
In this case we have $\theta = \tan^{-1}(-\sqrt{3}) = -60°$.

Principal values

We saw previously that $\sin 30° = \frac{1}{2}$.



But it is also true that $\sin 150° = \frac{1}{2}$



Also...$\sin 390° = \frac{1}{2}$ and $\sin(-210°) = \frac{1}{2}$.

There are an infinite number of angles θ for which $\sin\theta = \frac{1}{2}$.

Principal values of inverse trig functions

Your calculator does not list all the possible values of θ for an inverse trig function. It returns one principal value.

It is up to you to calculate the others if you need them.

The inverse functions are defined so that only one possible output (θ) can come from each input value.

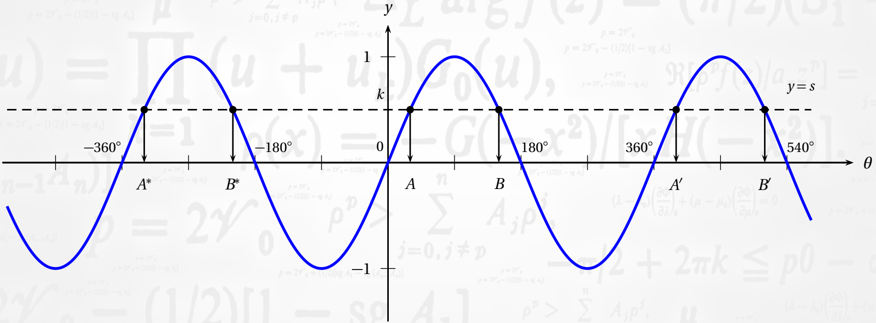
	PV	Range (°)	Range (rad)
$\sin x = s$	$\sin^{-1}(s)$	$-90° \le PV \le 90°$	$-\frac{\pi}{2} \le PV \le \frac{\pi}{2}$
$\cos x = c$	$\cos^{-1}(c)$	$0° \le PV \le 180°$	$0 \le PV \le \pi$
$\tan x = t$	$\tan^{-1}(t)$	$-90° < PV < 90°$	$-\frac{\pi}{2} < PV < \frac{\pi}{2}$

So, when $\sin x = \frac{1}{2}$ the principal value is $\sin^{-1}(0.5) = 30°$ and not any other angle.

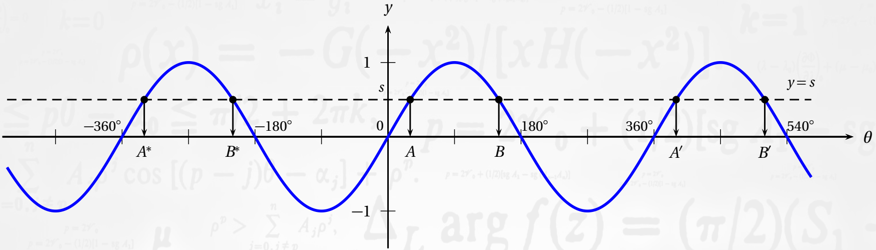
Solution of $\sin\theta = s$

The equation $\sin\theta = s$, where $-1 \leq s \leq 1$ has a solution in the interval $-90^\circ \leq x \leq 90^\circ$ called the *principal value* (PV).

On the graph of $y = \sin\theta$ the solutions we need are the places where the line $y = s$ intersects the curve $y = \sin\theta$.



The periodic nature of the sine curve means that we can calculate any of the solutions if we know the principal value.



Given the x-value of A (the principal value PV) then adding 360° gives the x-value of A'.
More solutions are found by adding (or subtracting) multiples of 360°

$$x = 360^\circ n + PV$$

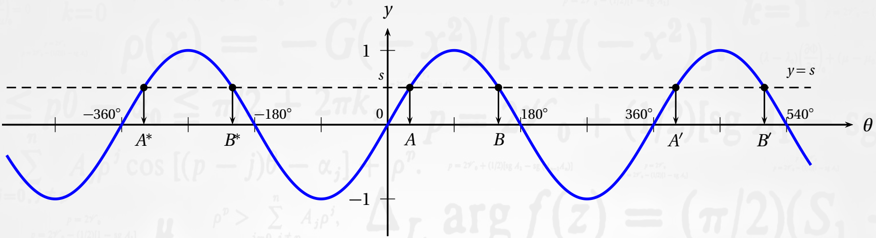
Where n is an integer.
This provides half of the required solutions.

The others are found using B as a starting point:

$$B = 180^\circ - A$$

B is called the secondary value (SV).
The remaining solutions to the equation come from

$$x = 360^\circ n + SV$$



To summarise — if we want all solutions to the equation

$$\sin\theta = s \qquad \text{where} \quad -1 \le s \le 1$$

First calculate the principal value

$$PV = \sin^{-1} s$$

and the secondary value

$$SV = 180^\circ - PV$$

The general solution is given by

$$\theta = \begin{cases} 360^\circ n + PV \\ 360^\circ n + SV \end{cases}$$

If radians are being used;

$$SV = \pi - PV$$

and

$$\theta = \begin{cases} 2n\pi + PV \\ 2n\pi + SV \end{cases}$$

Solving a trig equation with sine

Solve the equation $\sin\theta = \frac{1}{2}$ giving all solutions in the interval $0 < \theta < 720°$.

The principal value is

$$PV = \sin^{-1}(\tfrac{1}{2}) = 30°$$

The secondary value is given by $SV = 180 - 30 = 150°$.
The solutions to the equation are found from

$$\theta = \begin{cases} 360° n + 30° \\ 360° n + 150° \end{cases}$$

Now evaluate these with integer values of n until we find all solutions in $0 < \theta < 720°$.
With $n = 0$ we obtain the two trivial solutions:

$$\theta = 30°, \ \theta = 150°$$

With $n = 1$

$$\theta = 360(1) + 30 = 390°$$

and

$$\theta = 360(1) + 150 = 510°$$

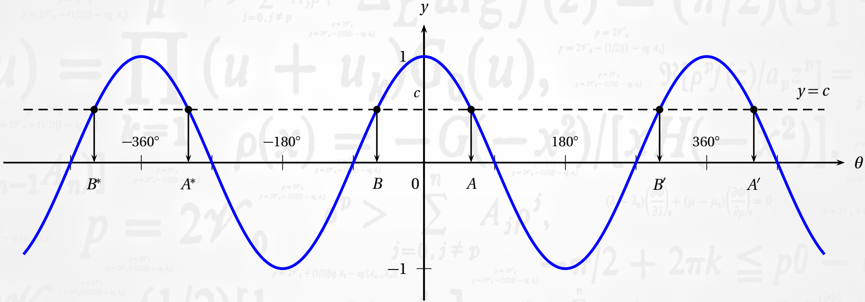
When we try $n = 2$ we get $\theta = 750°$ and $\theta = 870°$, which both lie *outside* the specified interval.
The set of solutions to the original equation are:

$$\theta = 30°, 150°, 390° \text{ and } 510°$$

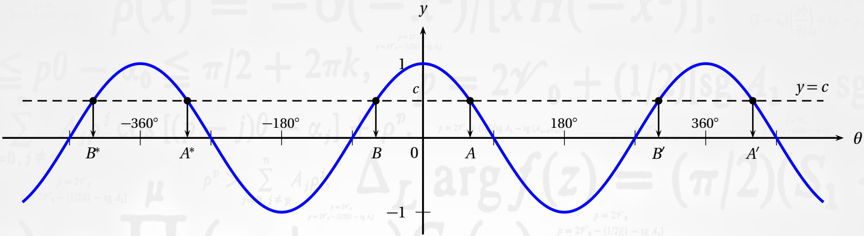
Solution of $\cos\theta = c$

The equation $\cos\theta = c$, where $-1 \le c \le 1$ has a solution in the interval $0° \le \theta \le 180°$ called the principal value (PV).

On the graph of $y = \cos\theta$ the complete set of solutions are the places where the line $y = c$ intersects the curve $y = \cos\theta$.



The periodic nature of the cosine curve means that we can calculate any of the solutions if we know the principal value.



Given the value of A (the principal value PV) then adding 360° gives the value of A'.

Other solutions are found by adding (or subtracting) multiples of 360° (subtracting gives A^*), so

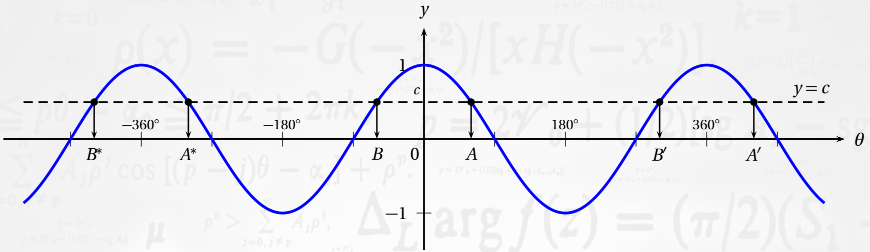
$$\theta = 360^\circ n + PV$$

Where n is an integer. The remaining solutions are found using B as a starting point:

$$B = -A = -PV$$

Therefore

$$\theta = 360^\circ n - PV$$



To summarise — if we want all solutions to the equation

$$\cos\theta = c \quad \text{where} \quad -1 \le c \le 1$$

First, calculate the principal value

$$PV = \cos^{-1} c$$

Since the secondary value is $-PV$ then the general solution is given by

$$\theta = 360^\circ n \pm PV$$

When radians are used we modify the formula to

$$\theta = 2n\pi \pm PV$$

where n is an integer.

Solving a trig equation with cosine

Solve $\cos\theta = -\frac{\sqrt{2}}{2}$, $-360° \le \theta \le 360°$.

The principal value (PV)

$$PV = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = 135°$$

The general solution for this problem is therefore given by:

$$\theta = 360° n \pm 135°$$

The particular solutions are obtained by trying different values of n.

With $n = 0$ we have

$$\theta = 135° \text{ and } \theta = -135°$$

With $n = 1$ we find:

$$\theta = 225° \text{ or } \theta = 495°$$

And we reject the second value as it lies outside of the interval.

With $n = -1$ we find

$$\theta = -225° \text{ or } \theta = -495°$$

Only the first of these is within the interval.

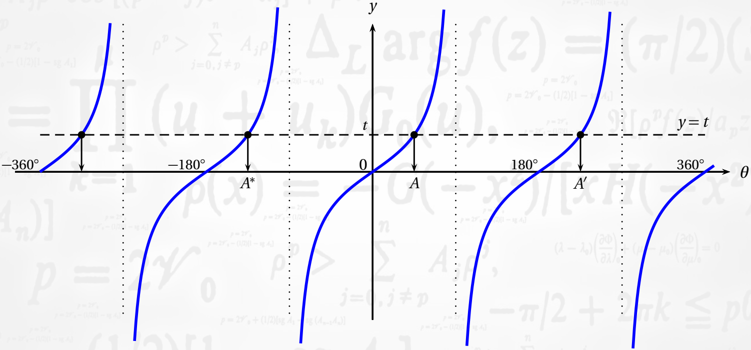
The set of solutions for this problem is therefore:

$$\theta = -225°, -135°, 135° \text{ and } 225°$$

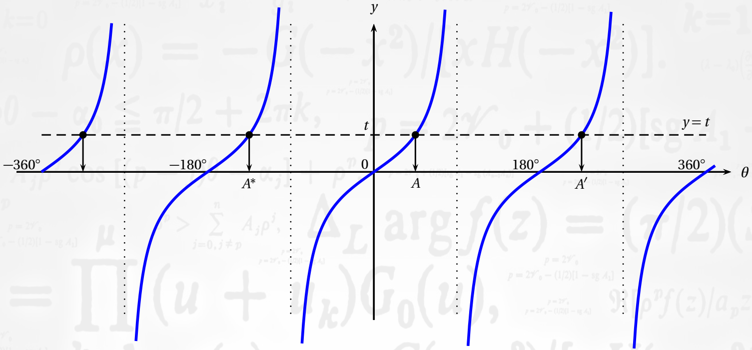
Solution of $\tan\theta = t$

The equation $\tan\theta = t$, where $-\infty < t < \infty$ has a solution in the interval $-90° < \theta < 90°$ called the principal value (*PV*).

Solutions to this equation are shown on the graph. Point *A* is the *PV*.



The periodic nature of the tangent graph means that we can calculate any of the solutions if we know the principal value.



The tangent graph is periodic so we just need to add (or subtract) multiples 180° to the principal value to obtain other solutions.

Therefore the general solution is:

$$\theta = 180^\circ n + PV$$

where $PV = \tan^{-1}(t)$. Or, if working with radians:

$$\theta = \pi n + PV$$

Where n is an integer.

Solving a trig equation with tangent

Solve $\tan\theta = -\sqrt{3}$, $-360^\circ \leq \theta \leq 360^\circ$

The principal value (PV)

$$PV = \tan^{-1}(-\sqrt{3}) = -60^\circ$$

The general solution for this problem is therefore given by:

$$\theta = 180n - 60$$

The particular solutions are obtained by trying different values of n.

With $n = 0$ we have the trivial solution $\theta = -60^\circ$.

With $n = 1$ we find:

$$\theta = 180(1) - 60 = 120^\circ$$

With $n = 2$

$$\theta = 180(2) - 60 = 300^\circ$$

$n = 3$ will not give useful values. Next we can try negative values of n.

With $n = -1$ we obtain:

$$\theta = 180(-1) - 60 = -240^\circ$$

And that's it — no other values of n will give θ values we need.

The solution set for this problem as:

$$\theta = -240^\circ, -60^\circ, 120^\circ \text{ and } 300^\circ$$

An alternative formula for sine

The general solution for sine was given by the two equations:

$$\theta = \begin{cases} 360^\circ n + PV \\ 360^\circ n + SV \end{cases}$$

where PV and SV are the principal and secondary values respectively.

It is possible combine these into a single equation:

$$\theta = (-1)^n PV + 180^\circ n$$

Or in radians:

$$\theta = (-1)^n PV + n\pi$$

The term $(-1)^n$ is like a switch which changes the sign of PV before adding it to multiples of 180° (or π rads).

Using the alternative sine formula

Solve $\sin x = 0.27$ to one decimal place in $-360^\circ < x < 360^\circ$.

The principal value (PV) is

$$PV = \sin^{-1}(0.27) = 15.7^\circ$$

The solutions to the equation are found from

$$x = (-1)^n (15.7) + 180^\circ n$$

Substitute values of n until all solutions in the interval are found.
With $n = 0$ we obtain the two trivial solutions:

$$x = (-1)^0 (15.7) + 0 = 15.7^\circ$$

With $n = 1$

$$x = (-1)^1 (15.7) + 180 = 165.3^\circ$$

Trying $n = 2$ gives

$$x = (-1)^2 (15.7) + 360 = 377.7^\circ$$

This is beyond the specified interval.
With $n = -1$

$$x = (-1)^{-1} (15.7) - 180 = -195.7^\circ$$

With $n = -2$

$$x = (-1)^{-2} (15.7) - 360 = -344.3^\circ$$

There are no further values to be found.
The solution set for the equation is therefore:

$$x = -344.3^\circ, -195.7^\circ, 15.7^\circ \text{ and } 165.3^\circ$$

Using radians

Solving a trig equation with tangent

Solve $\tan\theta = 1$, $-2\pi \le \theta \le 2\pi$

The interval, given in terms of π implies that angles are given in radians. The principal value (PV)

$$PV = \tan^{-1} 1 = \tfrac{\pi}{4}$$

The general solution for this problem is therefore given by:

$$\theta = \pi n + \frac{\pi}{4}$$

The particular solutions are obtained by trying different values of n.
With $n = 0$ we have the trivial solution $\theta = \frac{\pi}{4}$.
With $n = 1$ we find:

$$\theta = (1)\pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

With $n = 2$ the value of θ is greater than 2π. So next we can try negative values of n.
With $n = -1$ we obtain:

$$\theta = (-1)\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$$

With $n = -2$ we obtain another value in the interval:

$$\theta = (-2)\pi + \frac{\pi}{4} = -2\pi + \frac{\pi}{4} = -\frac{7\pi}{4}$$

There are no other values of θ in the interval so we can give the solution set for this problem as:

$$\theta = -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4} \text{ and } \frac{5\pi}{4}$$

General solution of trig equations

Here is a summary of all the results seen so far.

$$\sin\theta = k \quad \begin{cases} \theta = 360^\circ n + PV \text{ and } 360^\circ n + SV \\ \theta = 2\pi n + PV \text{ and } 2\pi n + SV \qquad n = 0, \pm 1, \pm 2, \ldots \end{cases}$$

where $PV = \sin^{-1} k$ and $SV = 180^\circ - PV$ (or $SV = \pi - PV$ in radians).

Or the alternative form for sine:

$$\sin\theta = k \quad \begin{cases} \theta = (-1)^n PV + \pi n & PV = \sin^{-1} k \\ \theta = (-1)^n PV + 180^\circ n & n = 0, \pm 1, \pm 2, \ldots \end{cases}$$

The general solutions for cosine and tangent:

$$\cos\theta = k \quad \begin{cases} \theta = 2\pi n \pm PV & PV = \cos^{-1} k \\ \theta = 360^\circ n \pm PV & n = 0, \pm 1, \pm 2, \ldots \end{cases}$$

$$\tan\theta = k \quad \begin{cases} \theta = \pi n + PV & PV = \tan^{-1} k \\ \theta = 180^\circ n + PV & n = 0, \pm 1, \pm 2, \ldots \end{cases}$$

Test yourself

Solve the following equations and give all solutions in the specified interval.

1. $\cos x = -\frac{\sqrt{2}}{2}$, $0 < x < 360°$.
2. $\tan x = \sqrt{3}$, $-180° < x < 360°$
3. $\sin x = 1$, $-180° < x < 360°$
4. $3\tan x = \sqrt{3}$, $0 < x < 2\pi$
5. $\cos x = \frac{1}{4}$, $-2\pi < x < 2\pi$. (Answer to 2 DP).

Answers:

1. $x = 135°$ and $225°$.
2. $x = -120°$, $60°$ and $240°$.
3. $x = -270°$ and $90°$.
4. $x = \frac{\pi}{6}$ and $\frac{7\pi}{6}$.
5. $x = -4.97$, -1.32, 1.32 and 4.97 rads.