$a\sin x + b\cos x$

Trigonometry 5

INU0115/515 (Maths 2)

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Objectives

The topic is still kinematics but before we proceed any further we'll need to brush up on some trig identities.

In this presentation we'll revise some concepts you've seen earlier:

- Compound angle identities
- Angles and quadrants

We'll go on an apply these concepts to the expression

$$a\sin x + b\cos x$$

which is basically a sum of sine waves of different amplitudes.

This is a pre-requisite for studying the final topic in our kinematics course: simple harmonic motion.

Compound angle identities

The following trigonometric identities will prove to be useful in the following analysis, so take some time to familiarise yourself with how they work.

$$\begin{aligned} \sin(A \pm B) &\equiv \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &\equiv \cos A \cos B \mp \sin A \sin B \end{aligned}$$

For example:

$$\begin{aligned} 10\sin(3x + 60^\circ) &\equiv 10(\sin 3x \cos 60^\circ + \cos 3x \sin 60^\circ) \\ &\equiv 10(\tfrac{1}{2}\sin 3x + \tfrac{\sqrt{3}}{2}\cos 3x) \\ &\equiv 5\sin 3x + 5\sqrt{3}\cos 3x \end{aligned}$$

The problem we'll have in this presentation is how to reverse the process.

Given the expression $5\sin 3x + 5\sqrt{3}\cos 3x$ how could we know to write it in the form $10\sin(3x + 60^\circ)$?

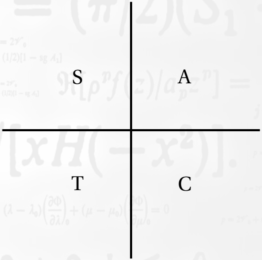
The four quadrants

The signs of the trig ratios sine, cosine and tangent take on different signs for angles between 0° and 360° (or 0 to 2π radians).

The sign changes in each quadrant are such that:

- [A] All ratios are positive in the first quadrant
- [S] Only sine is positive in the second quadrant
- [T] Only tangent is positive in the third quadrant
- [C] Only cosine is positive in the fourth quadrant

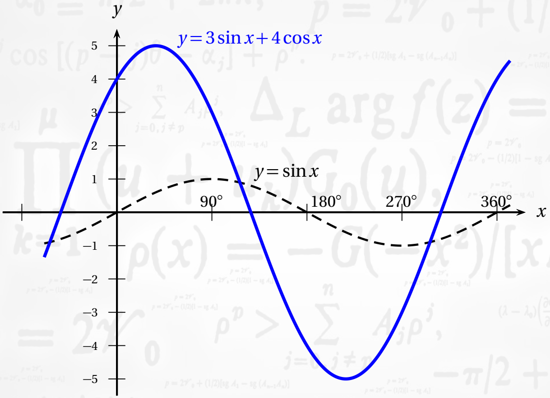
The capitalised letters refer to the quadrant diagram shown in the picture.



We've been considering quadrants in a lot of work recently: vectors, polar equations and complex numbers. We'll need it again soon for simple harmonic motion).

Combinations of sine and cosine

Consider the graph of the function $y = 3\sin x + 4\cos x$.



This curve has exactly the same shape as the curve $y = \sin x$ or $y = \cos x$ (i.e. it is a sine wave). In other words, we can apply transformations to stretch and shift the graphs of $y = \sin x$ or $y = \cos x$ to obtain this curve.

The expression $a\sin x + b\cos x$

Graphs of the function

$$y = a\sin x + b\cos x$$

can be produced by applying transformations to the sine or cosine function. This means we can write the expression $a\sin x + b\cos x$ in one of the following four ways:

$$R\sin(x+\alpha) \quad R\sin(x-\alpha) \quad R\cos(x+\alpha) \quad R\cos(x-\alpha)$$

In these expressions the positive coefficient R scales the amplitude (height) of the graph and α is an angle (degrees or radians) representing a horizontal shift along the x-axis. See your notes on Transformations from semester 1!

The purpose of changing the expression into one of these forms is that it is easier to analyse the behaviour of the function and to solve equations containing this expression.

Changing to a scaled, shifted sine function

Express the function $3\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 \le \alpha \le 360°$.

This is the example for which the graph was given earlier.

Begin by expanding $R\sin(x+\alpha)$ using correct compound angle identity:

$$R\sin(x+\alpha) \equiv R(\sin x\cos\alpha + \cos x\sin\alpha) \equiv R\cos\alpha\sin x + R\sin\alpha\cos x$$

We are looking for values of R and α which make this expression equal to $3\sin x + 4\cos x$ so we equate them:

$$3\sin x + 4\cos x \equiv (R\cos\alpha)\sin x + (R\sin\alpha)\cos x$$

The terms in brackets on the RHS are constant. For this identity to be true we need the same amount of $\sin x$ and $\cos x$ on both sides. Therefore:

$$R\cos\alpha = 3 \quad (1)$$

$$R\sin\alpha = 4 \quad (2)$$

Two equations with two unknowns. We can find R by squaring and adding the equations:

$$\begin{aligned} R^2\sin^2\alpha + R^2\cos^2\alpha &= 3^2 + 4^2 \\ R^2(\sin^2\alpha + \cos^2\alpha) &= 25 \\ R^2 &= 25 \end{aligned}$$

Therefore $R = 5$.

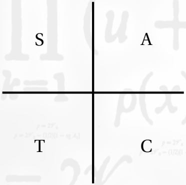
Changing to a scaled, shifted sine function

Express the function $3\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 \le \alpha \le 360°$.

We can find α by dividing equation (2) by equation (1).

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{4}{3} \quad \Rightarrow \tan\alpha = \frac{4}{3}$$

The principal value for this equation is $\alpha = \tan^{-1}\frac{4}{3} = 53.1°$ (to one decimal place).



A quick check: since equations (1) and (2) imply that $\sin\alpha$ and $\cos\alpha$ (and $\tan\alpha$) are all positive, so α must be an angle in the first quadrant (A - all positive).

Our value of $53.1°$ is in the first quadrant, so nothing further is required.

Therefore:

$$3\sin x + 4\cos x \equiv 5\sin(x + 53.1°)$$

The RHS can be obtained by stretching the $\sin x$ graph vertically by a factor of 5 and shifting it to the left by $53.1°$. This seems to be in agreement with graph seen earlier.

Express $y = \sin x - \cos x$ in the form $R\cos(x-\alpha)$, where $R > 0$ and $0 \le \alpha \le 2\pi$.

Begin by expanding $R\cos(x-\alpha)$ using the appropriate identity:

$$\begin{aligned} R\cos(x-\alpha) &\equiv R(\cos x \cos\alpha + \sin x \sin\alpha) \\ &\equiv R\cos\alpha\cos x + R\sin\alpha\sin x \end{aligned}$$

We are looking for values of R and α which make this expression equal to $\sin x - \cos x$.

Equate the two expressions:

$$\sin x - \cos x \equiv (R\cos\alpha)\cos x + (R\sin\alpha)\sin x$$

Again, the constant terms have been grouped. Compare the coefficients of $\sin x$ and $\cos x$ on both sides:

$$R\cos\alpha = -1 \qquad (3)$$

$$R\sin\alpha = 1 \qquad (4)$$

We can find R by squaring and adding the equations. That gives:

$$R^2 = 1^2 + (-1)^2 = 2 \qquad \therefore R = \sqrt{2}$$

Since $R > 0$ then $\sin\alpha$ is positive (equation 4) and $\cos\alpha$ is not, so α must be in the second quadrant.

We can calculate α by dividing the equations:

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{-1} \Rightarrow \tan\alpha = -1$$

The principal value for this equation is $\alpha = -\frac{\pi}{4}$

We add π to bring the angle to the correct quadrant so that $\alpha = \frac{3\pi}{4}$

Therefore:

$$\sin x - \cos x \equiv \sqrt{2}\cos(x - \tfrac{3\pi}{4})$$

Minimum and maximum values

Consider again the function

$$y = 3\sin x + 4\cos x$$

To find the values of x which give the minimum and maximum values for y is not obvious. But by changing the form:

$$y = 5\sin(x + 53.1^\circ)$$

then it becomes easier to find the minimum and maximum values.

The sine function has a maximum value at 90°. So our function will have a maximum value when

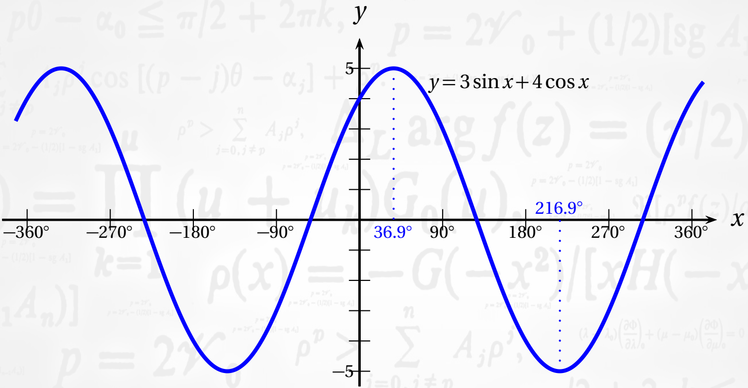
$$x + 53.1^\circ = 90^\circ \quad \therefore x = 36.9^\circ$$

The sine function has a minimum value at 270°. The function will have a minimum value when

$$x + 53.1^\circ = 270^\circ \quad \therefore x = 216.9^\circ$$

The function is periodic and so there will be other values (separated by multiples of 360° in this case).

Minimum and maximum values



The minimum and maximum values could also be determined by differentiation - see your semester 1 notes!

The equation $a\sin x + b\cos x = c$

The method for transforming the expression $a\sin x + b\cos x$ will allow us to solve any equation of the form

$$a\sin x + b\cos x = c \quad, |c| \leq \sqrt{a^2 + b^2}$$

Solving an equation

Solve the equation $4\cos x - \sqrt{20}\sin x = 3$ in the interval $0 \leq x \leq 360°$.

First, change the LHS of the equation into one of the four possible forms given by the compound angle identity. For example, the LHS of the equation can be expressed

$$6\cos(x - 311.81°) = 3$$

Solve this in the usual way:

$$\cos(x - 311.81°) = \tfrac{1}{2}$$

The principal value is 60° and the general solution is

$$x = 360n \pm 60° + 311.81°$$

The particular solutions are therefore $x = 11.81°, 251.81°$.

Application: Simple harmonic motion

A particle P moves in a straight line so that its distance x metres from an origin at time t seconds is given by:

$$x = 6\sin 2t - \cos 2t$$

Find the maximum distance of the particle to the right of the origin and the time when this first occurs.

We must express the function $x(t)$ in one of the four forms given earlier. The argument is $2t$ (instead of x) so we change the identity to show this.

For example, let's use $R\sin(2t-\alpha)$. Expand the identity to get

$$\begin{aligned} R\sin(2t-\alpha) &\equiv R(\sin 2t\cos\alpha - \cos 2t\sin\alpha) \\ &\equiv (R\cos\alpha)\sin 2t - (R\sin\alpha)\cos 2t \end{aligned}$$

Compare the coefficients with those of the x function to get:

$$R\sin\alpha = 1 \quad (5)$$

$$R\cos\alpha = 6 \quad (6)$$

The value of R^2 is obtained by squaring and adding these equations.

$$R^2 = 1^2 + 6^2 = 37 \quad \Rightarrow R = \sqrt{37}$$

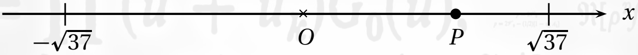
Equation (5) is divided by equation (6) to get

$$\tan\alpha = \frac{1}{6}$$

The principal value, in radians, is $\alpha = 0.1651$ (to 4 SF). A quick check of equations (5) and (6) shows that this is in the first quadrant as required. Therefore:

$$x = \sqrt{37}\sin(2t - 0.1651)$$

The amplitude of this function is $\sqrt{37}$ and this value represents the maximum distance of the particle from the origin.



The value of x is negative if the particle is to the left of the orgin, and positive when it is to the right.

Since we require the distance to the right then the maximum value of x must occur when $\sin(2t - 0.1651) = 1$.

We require $2t - 0.1651 = \frac{\pi}{2}$ to satisfy this equation. Rearrange to get $t = 0.87$ seconds.

The particle reaches a maximum distance of $\sqrt{37} \approx 6.1$ metres to the right of O after 0.87 seconds.

Test yourself...

Let's finish with a quick assessment of your knowledge.

1. Given $y = 6\sin x + 8\cos x$, calculate the amplitude of y.
2. Express $3\sin x - 6\cos x$ in the form $R\sin(x+\alpha)$ where α is given to the nearest degree.
3. Express $8\sin 2x - 3\cos 2x$ in the form $R\cos(2x-\alpha)$ where α is given to the nearest degree.
4. Solve the equation
$$10\cos\left(3t - \frac{2\pi}{3}\right) = 5$$
and find the smallest, positive value of t.
5. Consider the function
$$x = \sin 4t - 2\cos 4t$$
Calculate the maximum value of x and the first, positive value of t for which x is at maximum. Give t to 3 DP.

Answers:

1. Amplitude is 10.
2. $R = \sqrt{45} = 3\sqrt{5}$ and $\alpha = 297°$.
3. $R = \sqrt{73}$ and $\alpha = 111°$
4. $t = \frac{\pi}{9}$.
5. $x_{max} = \sqrt{5}$ and $t = 0.669$.