PROVING IDENTITIES

TRIGONOMETRY 4

INU0115/515 (MATHS 2)

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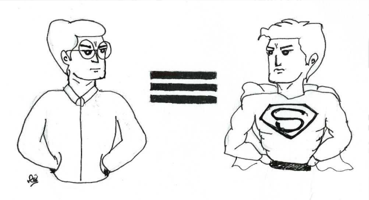
Proving an identity

Proving an identity is a process which starts with the LHS (left-hand side) or the RHS (right-hand side) of a given relationship and, through a series of logical steps, shows that the other side of the relationship can be obtained.



They may look different but the right side and left sides of an identity are the same.

Proving an identity

Prove the identity $(x+1)^3 \equiv x^3+3x^2+3x+1$

In this example, we can start with the LHS and make the following arguments.

$$\begin{aligned}
\text{LHS} &\equiv (x+1)^3 \\
&\equiv (x+1)(x+1)^2 \\
&\equiv (x+1)(x^2+2x+1) \\
&\equiv x(x^2+2x+1)+x^2+2x+1 \\
&\equiv x^3+2x^2+x+x^2+2x+1 \\
&\equiv x^3+3x^2+3x+1 \\
\therefore \text{LHS} &\equiv \text{RHS}
\end{aligned}$$

Proving a trig identity

Prove the identity $\sec\theta \equiv \tan\theta\sin\theta + \cos\theta$

Which side do we begin changing? We'll choose the RHS because there more things we can change.

$$\text{RHS} \equiv \tan\theta\sin\theta + \cos\theta$$

We know that $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ so we substitute this for $\tan\theta$:

$$\begin{aligned} \text{RHS} &\equiv \frac{\sin\theta}{\cos\theta}\sin\theta + \cos\theta \\ &\equiv \frac{\sin^2\theta}{\cos\theta} + \cos\theta \end{aligned}$$

Put the terms over a common denominator:

$$\text{RHS} \equiv \frac{\sin^2\theta + \cos^2\theta}{\cos\theta}$$

But know that $\sin^2\theta + \cos^2\theta \equiv 1$ so the top simplifies:

$$\begin{aligned} \text{RHS} &\equiv \frac{1}{\cos\theta} \\ &\equiv \sec\theta \\ \therefore \text{RHS} &\equiv \text{LHS} \end{aligned}$$

Proving a trig identity

Prove that $(1-\cos\alpha)(1+\sec\alpha) \equiv \sin\alpha\tan\alpha$.

Let's start with the LHS and make changes to it.

$$\begin{aligned}
\text{LHS} &\equiv (1-\cos\alpha)(1+\sec\alpha) \\
&\equiv 1+\sec\alpha-\cos\alpha-\cos\alpha\sec\alpha \\
&\equiv 1+\sec\alpha-\cos\alpha-\cos\alpha\left(\frac{1}{\cos\alpha}\right) \\
&\equiv 1+\sec\alpha-\cos\alpha-1 \\
&\equiv \sec\alpha-\cos\alpha
\end{aligned}$$

Remember — we are aiming to have an expression with $\sin\alpha$ and $\tan\alpha$.

$$\begin{aligned}
\text{LHS} &\equiv \frac{1}{\cos\alpha}-\cos\alpha \\
&\equiv \frac{1-\cos^2\alpha}{\cos\alpha} \\
&\equiv \frac{\sin^2\alpha}{\cos\alpha} \\
&\equiv \sin\alpha\left(\frac{\sin\alpha}{\cos\alpha}\right) \\
\text{LHS} &\equiv \sin\alpha\tan\alpha \\
\text{LHS} &\equiv \text{RHS}
\end{aligned}$$

Useful strategies...

There are no rules which for proving trig identities that will work in every situation. But we can talk about strategies or guidelines that might help.

- Start with more complicated side of the identity
- Start changing things!
  - e.g. $\tan\theta$ can be expressed as ${}^{\sin\theta}/_{\cos\theta}$.
  - e.g. $\tan^2\theta$ can be expressed as ${}^{\sin^2\theta}/_{\cos^2\theta}$ and so on.
  - Reciprocal trig ratios can be changed; e.g. $\sec\theta = {}^{1}/_{\cos\theta}$
- Look for the fundamental identity $\sin^2\theta + \cos^2\theta = 1$.
  - Or variations of it; $1 - \sin^2\theta$ can be replaced with $\cos^2\theta$.
- Look for Pythagorean identities; e.g. $\sec^2\theta - 1 = \tan^2\theta$.
- Be aware of other identities that might simplify things;
  - e.g. $\sin 2\theta$ can be expressed as $2\sin\theta\cos\theta$.
- If fractions are present — put them over a common denominator
- Cancel and simplify if possible.

This list is not exhaustive and we simply can't list all the ways you might prove an identity!

Proving a trig identity

Prove that $\dfrac{\tan x + \cot x}{\sec x \operatorname{cosec} x} \equiv 1$.

Starting with the LHS

$$\begin{aligned} \text{LHS} &\equiv \frac{\tan x + \cot x}{\sec x \operatorname{cosec} x} \\ &\equiv \frac{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}{\frac{1}{\cos x}\,\frac{1}{\sin x}} \end{aligned}$$

Multiply top and bottom by $\sin x \cos x$:

$$\begin{aligned} \text{LHS} &\equiv \frac{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}{\frac{1}{\sin x \cos x}} \times \frac{\sin x \cos x}{\sin x \cos x} \\ &\equiv \frac{\sin^2 x + \cos^2 x}{1} \\ &\equiv \frac{1}{1} \\ \text{LHS} &\equiv 1 \\ \text{LHS} &\equiv \text{RHS} \end{aligned}$$

Final comments...

In principle it's possible to start with either side of the identity and make changes until the other side has been reached.

$$\text{LHS} \Longrightarrow \text{RHS} \qquad \text{or} \quad \text{RHS} \Longrightarrow \text{LHS}$$

In practice it is often more obvious how to begin the proof with one side of the identity than it is to begin with the other side. This is something that gets easier to recognise through experience (doing more examples!)

Alternatively, we might begin with one side of the identity and change it a little. Then, perhaps we go to the other side of the identity and try to change it enough to 'meet halfway'.

$$\text{LHS} \Longrightarrow \quad \Longleftarrow \text{RHS}$$

This approach is like building a bridge from opposite sides of a river with the aim of meeting in the middle. If this works then the path from LHS to RHS (or vice versa) will be obvious at the end of the proof.

Finally — which results do we assume to be true to help us prove identities? For now, we can take the Pythagorean Identities to be proved since they were proved using a right-angled triangle in earlier in the course.