TRIG IDENTITIES

TRIGONOMETRY 4

INU0115/515 (MATHS 2)

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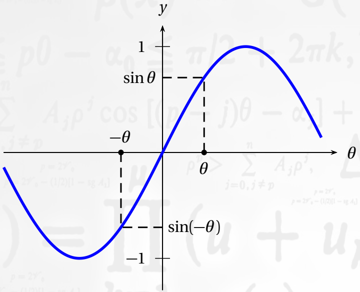
Objectives

This week we're building up our knowledge of trigonometric functions and graphs. It will serve us well when we're studying integral calculus in Semester 2!

In this presentation we'll study the following aspects of trig functions:

- Properties of sine and cosine.
- Compound angle identities
- Double angle identities
- Using identities to solve equations

Properties of sine and cosine



The picture shows part of the graph of $y = \sin\theta$.

For any angle θ we can obtain the value of $\sin\theta$.

Similarly, for the angle $-\theta$ we can find $\sin(-\theta)$.

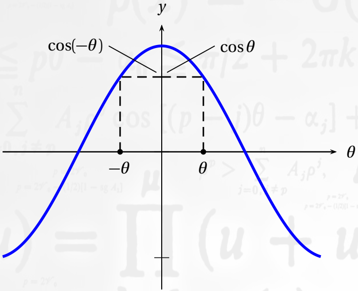
Mathematicians describe $\sin\theta$ as an odd function. Positive and negative domain values are related through:

$$\boxed{\sin(-\theta) \equiv -\sin\theta} \tag{1}$$

On the graph, the y-value of $\sin\theta$ is the same distance above the x-axis as $\sin(-\theta)$ is below the axis.

For example: $\sin(-30°) = -\frac{1}{2}$ and $-\sin(30°) = -\frac{1}{2}$.

Let's examine the cosine function now.



The picture here shows part of the graph of $y = \cos\theta$.

For any angle θ we can obtain the value of $\cos\theta$.

Similarly, for the angle $-\theta$ we can find $\cos(-\theta)$.

Mathematicians describe $\cos\theta$ as an even function. It means positive and negative domain values are related according to:

$$\boxed{\cos(-\theta) \equiv \cos\theta} \tag{2}$$

On the graph, the y-value of $\cos\theta$ is the same distance above the x-axis as $\cos(-\theta)$.

For example: $\cos(-60^\circ) = \frac{1}{2}$ and $\cos(60^\circ) = \frac{1}{2}$.

Compound angle identities

In problems where we have trig functions it is often useful to be able to express functions of compound angles such as

$$\sin(A+B) \quad \text{or} \; \cos(\theta-\alpha)$$

in terms of functions of the individual angles (e.g. $\sin A$ or $\cos\alpha$).

The following identities for sine, cosine and tangent often turn out to be useful in trig-related maths:

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B \qquad (3)$$
$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B \qquad (4)$$
$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad (5)$$

The proof of these identities can be found in your course textbook (Chapter 13, p257-9).

Using a compound identity

Express $\cos 75^\circ$ in surd form.

We can use the compound identity for cosine and write $A=45^\circ$ and $B=30^\circ$.

$$\begin{aligned} \cos 75^\circ &= \cos(45^\circ+30^\circ) \\ &= \cos 45 \cos 30 - \sin 30 \sin 45 \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \end{aligned}$$

Therefore $\cos 75^\circ = \frac{1}{4}(\sqrt{6}-\sqrt{2})$.

Double angle identities

Having derived compound angle identities (equations 3, 4 and 5) we will derive several more, which will prove to be useful in the future.

Setting $B = A = \theta$ in those identities we obtain the following:

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{6}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{7}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{8}$$

These are collectively called double angle identities.

Compound angle identity (again!)

Show that $\cos 3x \equiv 4\cos^3 x - 3\cos x$.

We can split the argument of the cosine function ($3x = 2x + x$) and then the cosine compound identity to rewrite it:

$$\begin{aligned} \cos 3x &\equiv \cos(2x + x) \\ &\equiv \cos 2x \cos x - \sin 2x \sin x \end{aligned}$$

We can further simplify this using the double angle identities

$$\begin{aligned} \cos 3x &\equiv (\cos^2 x - \sin^2 x)\cos x - (2\sin x \cos x)\sin x \\ &\equiv \cos^3 x - \sin^2 x \cos x - 2\sin^2 x \cos x \end{aligned}$$

We can use the identity $1 - \cos^2 x$ to simplify further:

$$\begin{aligned} \cos 3x &\equiv \cos^3 x - (1 - \cos^2 x)\cos x - 2(1 - \cos^2 x)\cos x \\ &\equiv \cos^3 x - \cos x + \cos^3 x - 2\cos x + 2\cos^3 x \\ \therefore \cos 3x &\equiv 4\cos^3 x - 3\cos x \end{aligned}$$

Identities and equations

Trig identities are frequently used to help solve equations. Usually, we aim to simplify an equation by replacing one or more terms using identities.

Use of double angle identities

Solve $\cos 2x + 5\sin x = 3$, $0° < x < 360°$.

We'll substitute the $\cos 2x$ term using the double angle identity $\cos 2x \equiv \cos^2 x - \sin^2 x$:

$$\cos^2 x - \sin^2 x + 5\sin x = 3$$

We can use the identity $\cos^2 x \equiv 1 - \sin^2 x$ to express the equation completely in terms of the sine function.

$$1 - \sin^2 x - \sin^2 x + 5\sin x = 3$$

And rearrange to get

$$2\sin^2 x - 5\sin x + 2 = 0$$

This is a quadratic equation in $\sin x$; factorise the quadratic to get

$$(2\sin x - 1)(\sin x - 2) = 0$$

So $\sin x = \frac{1}{2}$ and $\sin x = 2$.

The latter equation has no solutions but the former can be solved using our usual methods to give:

$$x = 30^\circ \text{ and } 150^\circ$$

Test yourself...

Use your knowledge of trig transformations to answer the following questions.

1. Given that $\sin 60° = \frac{\sqrt{3}}{2}$ what is the value of $\sin(-60°)$?
2. Given that $\cos 120° = -\frac{1}{2}$ what is the value of $\cos(-120°)$?
3. Express $\tan 75°$ in surd form.
4. Solve $\sin 2x + \sin x = 0$, $0 \leq x \leq 2\pi$.

Answers:

1. $\sin(-60°) = -\sin 60° = -\frac{\sqrt{3}}{2}$
2. $\cos(-120°) = \cos 120° = -\frac{1}{2}$
3. $\tan 75° = 2 + \sqrt{3}$.
4. $x = 0$, $\frac{2}{3}\pi$, π, $\frac{4}{3}\pi$, and 2π