Transformations

Trigonometry 4

INU0115/515 (Maths 2)

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Types of transformation

Given a trig function $f(x)$ we'll now examine what happens when it undergoes one or more of the following transformations:

1. $f(x) \to kf(x)$ where k is a constant.
   e.g. $\tan x$ changing to $5\tan x$.
2. $f(x) \to f(x)+k$, where k is a constant
   e.g. $\sin x$ changing to $5+\sin x$.
3. $f(x) \to f(x+\alpha)$ where α is an angle.
   e.g. $\tan x$ changing to $\tan(x-30°)$
4. $f(x) \to f(kx)$ where k is a constant.
   e.g. $\cos 2x$ changing to $\cos 2x$.

You can examine these changes yourself by visiting:

https://www.geogebra.org/m/b3UDrYuX

Terminology

Sine wave

A sine wave (also called a *sinusoid*) is a curve which varies smoothly and repetitively between a minimum and a maximum value.

Examples of sine waves are the graphs of $\sin x$ and $\cos x$.

Amplitude

The amplitude of a sine wave is the size of the maximum value of a sine wave. Amplitude is a positive quantity.

The graph of $\tan x$ does not have amplitude defined.

Period

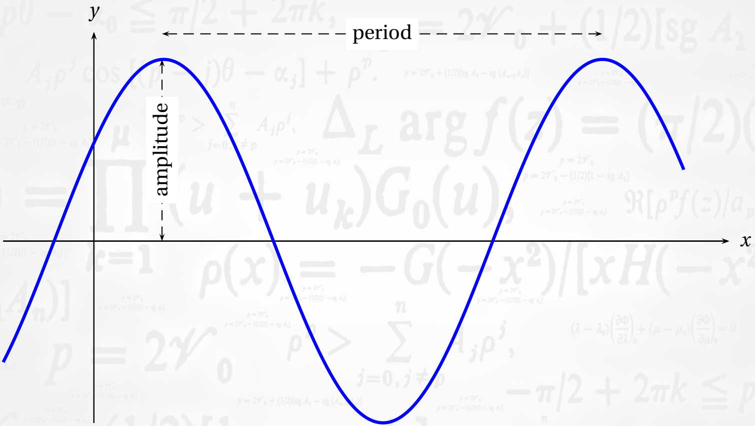
This is the angular 'distance' or time after which the function repeats. For example, it is the distance between two successive peaks of a cosine curve.

For example the period of the function $y = \tan x$ is $180°$.

Frequency

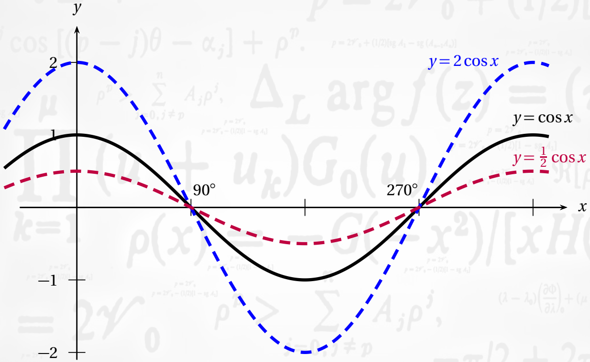
For a sine wave, the frequency is the number of cycles per unit interval (e.g. time).

Amplitude and period of a sine wave



Vertical scaling

The transformation $f(x) \rightarrow kf(x)$ has the effect of *stretching* or *compressing* the original graph in the vertical direction by a factor of k.

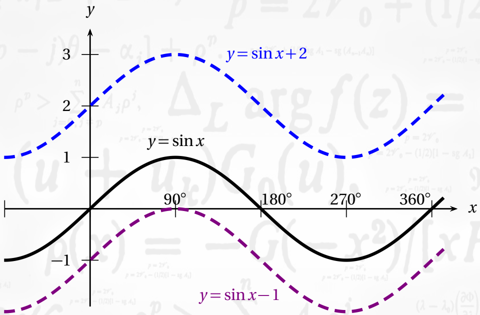


If the function is sine or cosine we describe this transformation in terms of the amplitude changing.

- When $k > 1$ the graph is *stretched* vertically
- When $0 < k < 1$ the graph is *compressed* vertically

Vertical translation

For the transformation $f(x) \to f(x) + k$ then the effect on the function graph is to shift it vertically by k. This type of change is also called a vertical translation.



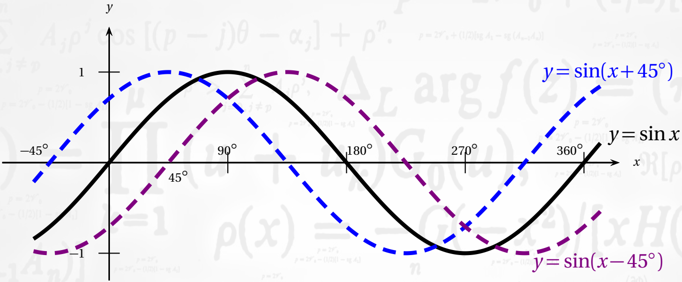
Vertical translations are shown for the function $f(x) = \sin x$. The graphs of two other sine functions with different values of k are also shown.

- When $k > 0$ the graph is shifted up.
- When $k < 0$ the graph is shifted down

The shape of graph is preserved; it is merely shifted up or down.

Horizontal translation

The transformation $f(x) \to f(x+\alpha)$, where α is an angle, causes a horizontal translation by an angle α. This type of transformation is also called a horizontal shift.



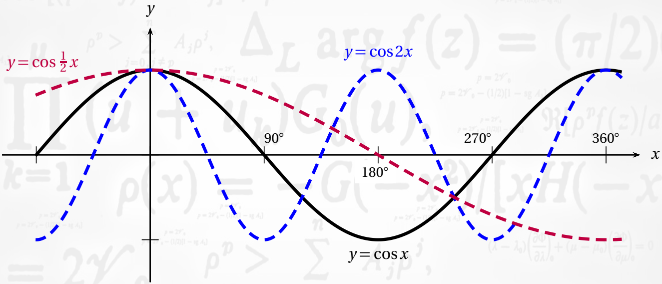
Horizontal translations are shown for the function $f(x) = \sin x$ in the graph. The graphs of two other functions with two different values of α are shown for comparison.

- When $\alpha > 0$ the graph is shifted to the left
- When $\alpha < 0$ the graph is shifted to the right

The shape of graph is preserved; it is merely left or right.

Horizontal scaling

The transformation $f(x) \to f(kx)$ causes a horizontal stretch or compression of the original graph.



- When $k > 1$ the graph is *compressed* horizontally
- When $0 < k < 1$ the graph is *stretched* horizontally

Some effects of a horizontal scaling:

- The period of sine and cosine function changes to

$$\text{Period} = \frac{360°}{k} \text{ or } \frac{2\pi}{k} \text{ rads}$$

- The period of the tangent function changes to

$$\text{Period} = \frac{180°}{k} \text{ or } \frac{\pi}{k} \text{ rads}$$

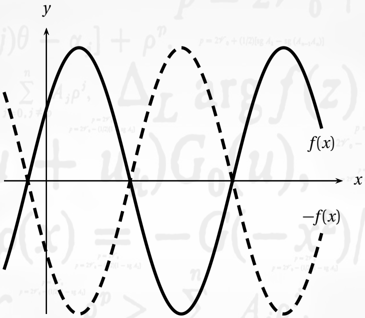
- $0 < k < 1$ (horizontal stretch)
  - the *frequency* has decreased by a factor of k, or
  - the *wavelength* has increased by a factor of k, or
  - the *period* has increased by a factor of k.
- $k > 1$ (horizontal compression)
  - The *frequency* has increased by a factor of k, or
  - the *wavelength* has decreased by factor of k, or
  - the *period* has decreased by a factor of k.

The tangent curve is not a wave. It only makes sense to describe the transformation in terms of its period.

When $k < 0$ the behaviour depends on the whether the function is odd or even.

Reflection in the x-axis

Vertical scalings $f(x) \to -kf(x)$ for $k < 0$ lead to a behaviour that we will examine now.

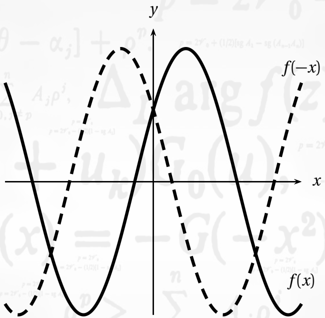


The transformation $f(x) \to -kf(x)$ produces a scaled reflection in the x-axis.

- When $k = -1$ the reflection is perfect — as shown in the picture above.
- For other negative values of k the reflection will have a different amplitude given by k.

Reflection in the y-axis

Horizontal scalings of the form $f(x) \to f(kx)$ with $k < 0$ lead to behaviour that we'll examine now.



The transformation $f(x) \to f(kx)$*, with* $k < 0$ *produces a scaled reflection in the y-axis.*

- When $k = -1$ the reflection is perfect — as shown in the figure above.
- For other negative values of k the reflection will have a different period (scaled by a factor of $1/k$).

Describing transformations

Describing a transformation

The function $\tan\theta$ is transformed to $3\tan\theta$. Describe this transformation in words.

This corresponds to a vertical stretch by a factor of 3

Describing a transformation

The function $\cos\theta$ is transformed to $\cos 2\theta$. Describe this transformation in words

This change corresponds to horizontal scaling. The argument has changed from θ to 2θ and so the graph is horizontally compressed by a factor of 2.

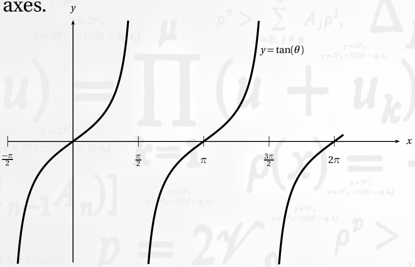
We could also describe this change in terms of the period decreasing by a factor of 2. Or by the frequency of the wave being doubled.

Graph sketching

Sketching a graph

Use the graph of $y = \tan\theta$ to sketch the graph of $y = \tan(\theta - \frac{1}{4}\pi)$

First sketch the graph of $\tan\theta$ and label the axes.

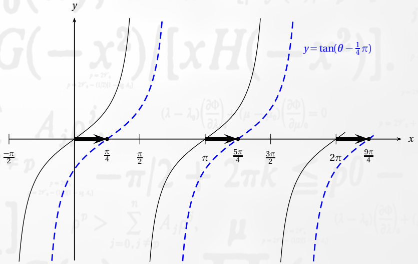


The transformation of $\tan\theta \to \tan(\theta - \frac{1}{4}\pi)$ is a horizontal shift: $\frac{1}{4}\pi$ radians to the right.

Any points where the original curve cuts across the x-axis move to the right.

Redraw the tangent curve through those points.

Other features such as asymptotes, although not shown here, also move.



Combining transformations

What happens when more than one type of transformation is applied to a graph?

The general case — when all the possible transformations seen previously are applied — will cause the function to change like this:

$$f(x) \to af(bx+\alpha)+c$$

where a, b, c and α are constants.

For example

- $\sin x$ changing to $4\sin 2x$.
- Or $\tan x$ changing to $2\tan(3x-45°)+1$.

We'll discuss how to describe these transformations and sketch the graphs of the transformed function using the original function as a basis.

It will be helpful if you review you work on composite functions from Maths 1.

Does the order of transformations matter?

Consider the transformations required to change $\sin x$ into $3\sin x + 2$.

Clearly there are two transformations at work;

- Vertical stretch by a factor of 3
- Vertical shift by 2.

In which order are they applied?

Starting with $f(x) = \sin x$ any transformation leads to a new function $g(x)$. And the next transformation acts on that function to make $h(x)$.

We're trying to arrange the transformations so that we end with $h(x) = 3\sin x + 2$.

Here are the two possible ways to apply the transformations:

Let $f(x) = \sin x$

(1) Vertical shift:

$$\begin{aligned} g(x) &= f(x) + 2 \\ &= \sin x + 2 \end{aligned}$$

(2) Vertical stretch:

$$\begin{aligned} h(x) &= 3g(x) \\ &= 3(\sin x + 2) \\ &= 3\sin x + 6 \end{aligned}$$

Let $f(x) = \sin x$

(1) Vertical stretch:

$$\begin{aligned} g(x) &= 3f(x) \\ &= 3\sin x \end{aligned}$$

(2) Vertical shift:

$$\begin{aligned} h(x) &= g(x) + 2 \\ &= 3\sin x + 2 \end{aligned}$$

The order matters!

Shifting up by 2 then scaling by a factor of 3 was *not* correct.

However, scaling by a factor of 3 and then shifting up by 2 units was correct.

Combining transformations

Describe the transformation $\cos x \to \cos(2x - 30°)$.

The two transformations at work here are *horizontal scaling* and *horizontal shifting*.

Let $f(x) = \cos x$
Horizontal shift:

$$\begin{aligned} g(x) &= f(x-30°) \\ &= \cos(x-30°) \end{aligned}$$

Horizontal compression:

$$\begin{aligned} h(x) &= g(2x) \\ &= \cos(2x-30°) \end{aligned}$$

Let $f(x) = \cos x$. Horizontal scaling:

$$g(x) = f(2x) = \cos(2x)$$

To get the correct function at the end, we'll have to make the horizontal shift 15° (not 30°).
Horizontal scaling:

$$\begin{aligned} h(x) &= g(x-15°) \\ &= \cos(2(x-15°)) \\ &= \cos(2x-30°) \end{aligned}$$

To summarise: if we want to transform $\cos x$ into $\cos(2x - 30°)$ the order we apply the transformations makes a difference.

We either:

- Shift to the right by 30° *then* compress by a factor of 2, or
- Compress by a factor of 2 *then* shift to the right by 15°.

Quick method

Is there a quicker way we can figure out just one correct sequence of transformations?

Yes — we can rewrite the original function to allow us to pick out the transformations at a glance. This is useful when we want to sketch a graph.

Given the function

$$y = af(bx + c) + d$$

Factorise the argument of the function into the form $b(x + \frac{c}{b})$ so that

$$\boxed{y = af\left[b\left(x + \frac{c}{b}\right)\right] + d}$$

After factorising we note the following:

1. Function will be vertically scaled by a factor of a
2. The period of the function is scaled by a factor $\frac{1}{b}$.
3. Horizontally shift the graph by an angle $\frac{c}{b}$. This is called the phase angle.
4. Vertically shift the graph by d units.

Combining transformations

Describe the transformations needed to change the graph of $y = \cos x$ into the graph

$$y = 4\cos(\tfrac{1}{2}x + \tfrac{\pi}{2}) - 1$$

We'll rewrite the function as

$$y = 4\cos\left[\tfrac{1}{2}(x + \pi)\right] - 1$$

and we note the following features about the transformed function:

1. The amplitude is 4 (vertical stretch)
2. The new period is $\frac{2\pi}{1/2} = 4\pi$ (horizontal stretch)
3. The phase shift is π radians to the left (horizontal translation)
4. Vertical translation (shift down) by 1 unit.

Test yourself...

Use your knowledge of trig transformations to answer the following questions.

1. What is the period, in degrees, of $y = 3\sin 4x$?
2. What is the amplitude of the function $f(x) = 5 + 2\cos(x - 45°)$?
3. Describe how the function $y = -\sin 2x$ can be obtained from $y = \sin x$.
4. Describe how the graph of $y = 3\tan(4x - \frac{\pi}{3})$ can be obtained from $y = \tan x$.

Answers:

1. Period is 90°.
2. Amplitude is 2.
3. e.g. Reflection in x-axis then decrease period by a factor of 2.
4. Vertical scale by factor of 3. Horizontal shift of $\frac{\pi}{12}$ (to the right). Period reduced by factor of 4.