Pythagorean Identities

Trigonometry 3

INU0115/515 (Maths 2)

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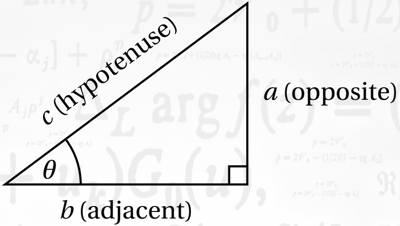
Objectives

In this presentation we'll cover the following topics:

- Derive some trig identities based on Pythagoras' theorem. They are called Pythagorean identities.
- Given an exact trig ratio we'll see how to derive ratios for other trig functions.
- Use Pythagorean identities to simplify and solve trig equations.

Recap: Pythagoras' Theorem

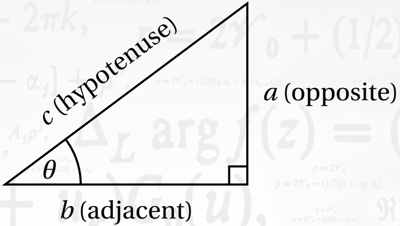


Consider the right-angled triangle *ABC* shown above.

A relationship exists between the lengths of the sides of a right-angled triangle; it is known as Pythagoras' Theorem and is represented by:

$$\boxed{a^2 + b^2 = c^2}$$

Recap: Basic trig ratios



Trigonometric ratios, defined for the angle θ are called sine, cosine and tangent (usually shortened to “sin”, “cos” and “tan” respectively).

With reference to the picture, these ratios are:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{a}{c} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{b}{c} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{a}{b}$$

Motivating example

Quite often in mathematics we are given a quantity and need to express it differently to make progess.

In trigonometry we might be given a trig ratio such as

$$\sin\theta = \tfrac{3}{7}$$

How can we calculate the value of $\tan\theta$?

A naïve way forward might be to find

$$\theta = \sin^{-1}(\tfrac{3}{7}) = 25.3769335^\circ \text{ (to 7 D.P.)}$$

so that

$$\tan\theta = \tan(25.3769335^\circ) = 0.4743416 \text{ (to 7 D.P.)}$$

But the value for $\tan\theta$ is no longer exact.

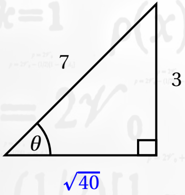
How can we calculate an exact ratio for $\tan\theta$?

Exact trig ratios

We can use Pythagoras' theorem to obtain exact trig ratios.

For example, given $\sin\theta = \frac{3}{7}$ we can construct a right-angle triangle where $\frac{\text{opp}}{\text{hyp}} = \frac{3}{7}$.

The triangle looks like this:



The length of the missing side was found using Pythagoras' theorem: $7^2 - 3^2 = 40$, so the missing side is $\sqrt{40}$.

Since all the sides are now known then the exact values $\tan\theta$ can also be found: using the definitions given earlier, can be found:

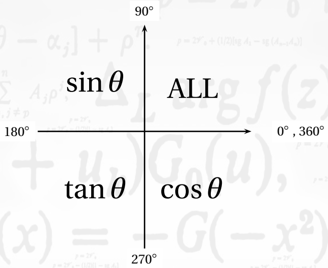
$$\tan\theta = \frac{3}{\sqrt{40}}$$

You can check that $\frac{3}{\sqrt{40}} \approx 0.4743416$, which is what we found on the previous slide.

Acute and obtuse angles

In the previous example we started with $\sin\theta = \frac{3}{7}$ and sketched a triangle.

In the sketch we assumed the angle θ *was an acute angle.*



Recall this diagram. It shows which trig ratios are *positive* in each quadrant.

Given that $\sin\theta = \frac{3}{7}$ then θ could be in the 1st or 2nd quadrant.

- If θ is acute (first quadrant) then $\tan\theta$ is positive.
- If θ is obtuse (second quadrant) then $\tan\theta$ is negative.

Therefore we should write: $\tan\theta = \pm\dfrac{3}{\sqrt{40}}$

Identities

Definitition of an identity

An identity is an equation which is true for all values of the variable involved. To emphasise this relationship the identity symbol $\equiv$ should be used.

Here is an identity:

$$(x+2)^2 \quad \equiv \quad x^2+4x+4$$

Any value of x substituted in the LHS will give the same value if it is substituted into the RHS. For example $x=0$ produces 4 on both sides.

Identities cannot be rearranged and solved like an equation!

Next we will derive some identities using trig functions.

Fundamental trig identity

We already know that:

$$\sin\theta = \frac{a}{c} \quad \cos\theta = \frac{b}{c} \quad \tan\theta = \frac{a}{b}$$

Dividing the ratios for sine and cosine gives:

$$\frac{\sin\theta}{\cos\theta} = \frac{a/c}{b/c} = \frac{a}{b}$$

But we already saw that $\tan\theta = a/b$, so therefore:

$$\boxed{\tan\theta \equiv \frac{\sin\theta}{\cos\theta}} \qquad (1)$$

This is an identity because it's true for all values of θ.

There's an interesting connection between the squares of $\sin\theta$ and $\cos\theta$:

$$(\sin\theta)^2 + (\cos\theta)^2 = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$$
$$= \frac{a^2}{c^2} + \frac{b^2}{c^2}$$
$$= \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1$$

$a^2 + b^2 = c^2$ by Pythagoras' theorem. Therefore:

$$\boxed{\sin^2\theta + \cos^2\theta \equiv 1}$$

Note that $\sin^2\theta$ means $(\sin\theta)^2$ in this identity.

This is often referred to as the *fundamental trig identity*.

Pythagorean identities

Starting with the fundamental trig identity

$$\boxed{\sin^2\theta + \cos^2\theta \equiv 1} \qquad (2)$$

If we divide both sides by $\cos^2\theta$ we get:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$$

We can simplify the second term on the LHS to 1. The first term $\left(\frac{\sin\theta}{\cos\theta}\right)^2$ simplifies to $\tan^2\theta$ using the first identity (1).

The term on the RHS is $\left(\frac{1}{\cos\theta}\right)^2$ which is $\sec^2\theta$. So now we have:

$$\boxed{\tan^2\theta + 1 \equiv \sec^2\theta} \qquad (3)$$

Returning again to identity (2) and dividing both sides by by $\sin^2\theta$ we get:

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} \equiv \frac{1}{\sin^2\theta}$$

This can be simplified for similar reasons to

$$\boxed{1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta} \qquad (4)$$

The identities (1) to (4) will prove useful in solving more complicated trigonometric equations.

Exact trig ratios (again!)

Trig identities give us another way to evaluate trig ratios exactly. They also provide a little more understanding of the earlier method.

Exact ratios

Given that $\sin x = \frac{1}{2}$ and x is obtuse, find $\cos x$.

Using $\sin^2 x + \cos^2 x \equiv 1$ we can rearrange and write:

$$\cos^2 x \equiv 1 - \sin^2 x$$

And therefore

$$\cos^2 x = 1 - (\tfrac{1}{2})^2 = 1 - \tfrac{1}{4} = \tfrac{3}{4}$$

Now if $\cos^2 x = \frac{3}{4}$ then there are *two* values for $\cos x$ — the positive and negative root:

$$\cos x = \pm \frac{\sqrt{3}}{2}$$

Referring to the earlier quadrant diagram, we note that in the 2nd quadrant (obtuse x) then cosine is negative, so we choose the second value

$$\cos x = -\frac{\sqrt{3}}{2}$$

Simplifying trig equations

Using identities to solve equations

Solve the equation

$$2\sin^2 x - \cos x - 1 = 0$$

giving all solutions in the interval $0 \le x \le 360°$.

The equation contains a mixture of sine and cosine; so simplify using identity 2.

Rearrange it as $\sin^2 x = 1 - \cos^2 x$. Then substitute the $\sin^2 x$ term in the equation to get:

$$2(1 - \cos^2 x) - \cos x - 1 = 0$$

Expand the brackets and simplify again:

$$\begin{aligned} 2 - 2\cos^2 x - \cos x - 1 &= 0 \\ -2\cos^2 x - \cos x + 1 &= 0 \\ 2\cos^2 x + \cos x - 1 &= 0 \end{aligned}$$

This is a quadratic equation in $\cos x$. Solve by factorising to get

$$(2\cos x - 1)(\cos x + 1) = 0$$

(Or use the quadratic formula!)

Now we must solve $\cos x = \frac{1}{2}$ and $\cos x = -1$ in the interval $0 \le x \le 360°$.

The first equation gives solutions $x = 60°$ and $x = 300°$.

The second equation has only one solution in the interval: $x = 180°$.

So the original equation has solutions $x = 60°, 180°, 300°$.

Using identities to solve equations

Solve the equation

$$10\,\text{cosec}^2 x = 16 - 11\cot x$$

giving all solutions in the interval $0 \le x \le 2\pi$.

We see a mixture of cosec and cot functions so simplify using identity 4.

Substitute $\text{cosec}^2 x \equiv 1 + \cot^2 x$ on the LHS:

$$10(1 + \cot^2 x) = 16 - 11\cot x$$

Expand the brackets and bring all the terms to one side:

$$10 + 10\cot^2 x = 16 - 11\cot x$$

$$10\cot^2 x + 11\cot x - 6 = 0$$

This is a quadratic equation in $\cot x$.

It can be factorised:

$$(2\cot x + 3)(5\cot x - 2) = 0$$

The solutions to this are

$$\cot x = -\tfrac{3}{2} \text{ and } \cot x = \tfrac{2}{5}$$

Change these to

$$\tan x = -\tfrac{2}{3} \text{ and } \tan x = \tfrac{5}{2}$$

The first equation has principal value of –0.588 rads, which gives solutions $x = 2.554$ rads and $x = 5.695$ rads.

Solving the second equation gives $x = 1.190$ rads and $x = 4.332$ rads.

So the complete set of solutions: $x = 1.190^c, 2.554^c, 4.332^c, 5.695^c$.

Test yourself...

Use your knowledge to answer the following questions.

1. Given that $\cos x = \frac{1}{5}$ find $\sin x$.
2. Given $\tan x = \frac{9}{4}$ find $\sec x$.
3. Given $\sin x = \frac{3}{4}$ and that x is obtuse, find $\cos x$.
4. Solve $\sec^2 x - 3\tan x - 5 = 0$, $0 < x < 360°$. (Answers to 1 DP).

Answers:

1. $\sin x = \pm\frac{2\sqrt{6}}{5}$ (or $\pm\frac{\sqrt{24}}{5}$)
2. $\sec x = \pm\frac{\sqrt{97}}{4}$.
3. $\cos x = -\frac{\sqrt{7}}{4}$
4. (Use $\sec^2 x \equiv 1 + \tan^2 x$)
   $x = 76.0°$, $135°$, $256.0°$ and $315°$.