Partial Fractions

Series 4

INU0114/514 (Maths 1)

Dr Adrian Jannetta MIMA CMath FRAS

INTO Newcastle University

Proper and Improper Algebraic Fractions

An algebraic fraction is said to be *proper* if the degree of the *numerator* is less than that of the *denominator*. Some examples of proper fractions are:

$$\frac{2}{(x+3)(x+5)} \qquad \frac{x}{x^2-5x+6} \qquad \frac{x^2-2x+1}{(x+3)(x-2)(x-7)}$$

An algebraic fraction is said to be *improper* if the degree of the *numerator* is greater than or equal to the degree of the *denominator*.

$$\frac{x^2}{(x+3)(x+5)} \qquad \frac{2x^4}{x^2-5x+6} \qquad \frac{x^3-2x+1}{(x+3)(x-2)(x-7)}$$

Improper fractions can be expressed using proper fractions by carrying out a process of polynomial division.

Algebraic Fractions

In semester 1 we looked at algebraic fractions and, in particular we studied:

- Cancelling common factors.
- Multiplying or dividing algebraic fractions.
- Adding or subtracting algebraic fractions.

We will briefly review how we add or subtract algebraic fractions before discussing how these are related to partial fractions.

Recap: Algebraic fractions

The method for adding or subtracting algebraic fractions is the same as for normal fractions.

1. Create a common denominator.
2. Adjust the numerators of each fraction.
3. Simplify the expression.

Adding Algebraic Fractions

Simplify the expression $\dfrac{1}{x+2}+\dfrac{2}{x-3}$.

$$\begin{aligned}\frac{1}{x+2}+\frac{2}{x-3} &\equiv \frac{1(x-3)+2(x+2)}{(x+2)(x-3)}\\ &\equiv \frac{x-3+2x+4}{(x+2)(x-3)}\\ &\equiv \frac{3x+1}{(x+2)(x-3)}\end{aligned}$$

Partial Fractions

- To generate partial fractions we reverse the process of adding or subtracting algebraic fractions.
- For a *proper fraction* we can begin to split the fraction into partial fractions.
- For *improper fractions* we have to do some work, usually polynomial division, to express it as a proper fraction.

Linear Factors

An algebraic fraction in which the denominator only contains factors of the form $ax+b$ where a and b are constants, are described as *linear factors.*

Express $\dfrac{6x-8}{(x-1)(x-2)}$ in terms of partial fractions.

We can write this as two partial fractions over linear factors:

$$\frac{6x-8}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2}$$

where A and B are constants that we must find.
Combine the partial fractions on the RHS into a single expression with a common denominator:

$$\frac{6x-8}{(x-1)(x-2)} \equiv \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

$$\frac{6x-8}{(x-1)(x-2)} \equiv \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

The LHS is identical to the RHS, and the denominators are identical, so the numerators must be identical:

$$6x-8 \equiv A(x-2)+B(x-1)$$

We'll choose values of x to easily determine A and B.
Choose $x=1$ (because it cancels the term with B:

$$\begin{aligned} 6(1)-8 &= A(1-2)+B(1-1) \\ -2 &= A(-1) \\ -2 &= -A \\ \therefore A &= 2 \end{aligned}$$

Choose $x = 2$:

$$\begin{aligned} 4 &= A(0) + B(1) \\ 4 &= B \\ \therefore B &= 4 \end{aligned}$$

The partial fractions are:

$$\frac{6x-8}{(x-1)(x-2)} \equiv \frac{2}{x-1} + \frac{4}{x-2}$$

Split $\dfrac{x+9}{x^2+9x+18}$ into partial fractions.

The denominator will factorise:

$$\frac{x+9}{x^2+9x+18} \equiv \frac{x+9}{(x+3)(x+6)} \equiv \frac{A}{x+3} + \frac{B}{x+6}$$

where A and B are constants.
From this we obtain:

$$x+9 \equiv A(x+6) + B(x+3)$$

Choosing $x = -6$ leads to $B = -1$.
Choosing $x = -3$ leads to $A = 2$.
Therefore

$$\frac{x+9}{x^2+9x+18} \equiv \frac{2}{x+3} - \frac{1}{x+6}$$

Decompose $\dfrac{2x^2+5x-13}{(x-1)(x+2)(x-3)}$ into partial fractions.

In this case our solution will contain 3 partial fractions:

$$\frac{2x^2+5x-13}{(x-1)(x+2)(x-3)} \equiv \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$$

Creating a common denominator for the RHS gives:

$$\frac{2x^2+5x-13}{(x-1)(x+2)(x-3)} \equiv \frac{A(x+2)(x-3)+B(x-1)(x-3)+C(x-1)(x+2)}{(x-1)(x+2)(x-3)}$$

The numerators must be equal so:

$$2x^2+5x-13 \equiv A(x+2)(x-3)+B(x-1)(x-3)+C(x-1)(x+2)$$

Choose the values of x to help determine the values of A, B and C.

$$x=1: \qquad -6=-6A \qquad \therefore A=1$$

$$x=-2: \qquad -15=15B \qquad \therefore B=-1$$

$$x=3: \qquad 20=10C \qquad \therefore C=2$$

Therefore

$$\frac{2x^2+5x-13}{(x-1)(x+2)(x-3)} \equiv \frac{1}{x-1} - \frac{1}{x+2} + \frac{2}{x-3}$$

Repeated Factors

Consider the following algebraic fractions:

$$\frac{x}{(x+2)^2} \qquad \frac{x}{(x+1)(x-2)(x-2)} \qquad \frac{1}{(x+3)(x-1)^2}$$

We have repeated factors in the denominator.

How could we separate $\frac{x}{(x+2)^2}$ into partial fractions?

If we use the same method as before we get:

$$\frac{x}{(x+2)^2} \equiv \frac{x}{(x+2)(x+2)} \equiv \frac{A}{(x+2)} + \frac{B}{(x+2)}$$

$$\frac{x}{(x+2)^2} \equiv \frac{A}{(x+2)} + \frac{B}{(x+2)} \equiv \frac{A+B}{x+2} \equiv \frac{C}{x+2}$$

So we have:

$$\frac{x}{(x+2)^2} \equiv \frac{C}{x+2}$$

Clearly LHS $\neq$ RHS so our method for splitting into partial fractions was clearly wrong.

If the algebraic fraction has a repeated factor in the denominator then the partial fractions must contain terms with a repeated factor *and* a linear factor.

$$\frac{x}{(x+2)^2} \equiv \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

Creating a common denominator for the RHS gives:

$$\frac{x}{(x+2)^2} \equiv \frac{A(x+2)+B}{(x+2)^2}$$

Because the denominators are the same we can write

$$x \equiv A(x+2)+B$$

$$x \equiv A(x+2) + B$$

Choose the values of x to help determine the values of A and B.
Put $x = -2$ to show $B = -2$.
We can use any value for x now.
Let's take $x = 0$:

$$\begin{aligned} 2 &= 2A \\ \therefore A &= 1 \end{aligned}$$

Therefore:

$$\frac{x}{(x+2)^2} \equiv \frac{1}{x+2} - \frac{2}{(x+2)^2}$$

Express $\frac{6x-8}{(x-1)(x-2)^2}$ as partial fractions.

Here we have one linear term and a repeated term so the partial fractions will have the form:

$$\frac{6x-8}{(x-1)(x-2)^2} \equiv \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

Creating a common denominator for the RHS gives:

$$\frac{6x-8}{(x-1)(x-2)^2} \equiv \frac{A(x-2)^2 + B(x-2)(x-1) + C(x-1)}{(x-1)(x-2)^2}$$

And so

$$6x-8 \equiv A(x-2)^2 + B(x-2)(x-1) + C(x-1)$$

$$6x-8 \equiv A(x-2)^2 + B(x-2)(x-1) + C(x-1)$$

Choose appropriate values of x to find A, B and C.
$x=1$:

$$\begin{aligned} -2 &= A \\ \therefore A &= -2 \end{aligned}$$

$x=2$ leads to $C=4$.
We can use any value for x now, so let's take $x=0$:

$$\begin{aligned} -8 &= 4A+2B-C \\ -8 &= 4(-2)+2B-4 \\ 4 &= 2B \\ \therefore B &= 2 \end{aligned}$$

Therefore

$$\frac{6x-8}{(x-1)(x-2)^2} \equiv -\frac{2}{x-1} + \frac{2}{x-2} + \frac{4}{(x-2)^2}$$

Irreducible Quadratic Factors

- An *irreducible quadratic* factor is a quadratic that cannot be factorised, such as x^2+3.
- If there is an irreducible quadratic in the denominator we must include a linear term in the numerator of the partial fractions.

Express $\dfrac{3x-1}{(x+1)(x^2+3)}$ as partial fractions.

There is a linear factor and an irreducible quadratic. The RHS will have two terms:

$$\frac{3x-1}{(x+1)(x^2+3)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+3}$$

Create a common denominator on the RHS:

$$\frac{3x-1}{(x+1)(x^2+3)} = \frac{A\left(x^2+3\right)+(Bx+C)(x+1)}{(x+1)(x^2+3)}$$

Therefore

$$3x-1 = A(x^2+3)+(Bx+C)(x+1)$$

Choose $x=-1$:

$$-4=4A \qquad \therefore A=-1$$

Choose $x=0$:

$$\begin{aligned} -1 &= 3A+B(0)+C \\ -1 &= \therefore C-3+C \qquad \therefore C=2 \end{aligned}$$

Put $x=1$ (we can choose any value):

$$\begin{aligned} 2 &= 4A+(B+C)(2) \\ 2 &= -4+2B+4 \\ 2 &= 2B \qquad \therefore B=1 \end{aligned}$$

And so

$$\frac{3x-1}{(x+1)(x^2+3)} = \frac{-1}{x+1}+\frac{x+2}{x^2+3}$$

Improper Fractions

An *Improper Fraction* is one where the numerator has a degree (highest power in x) equal to or greater than the denominator.

- Perform polynomial division to obtain a quotient and a remainder.
- Split the remainder into partial fractions.

Express $\dfrac{3x^2-3x-2}{x^2-3x+2}$ in terms of partial fractions.

Use polynomial division to simplify:

$$\frac{3x^2-3x-2}{x^2-3x+2} \equiv 3+\frac{6x-8}{x^2-3x+2} \equiv 3+\frac{6x-8}{(x-1)(x-2)}$$

Now, we need to split $\frac{6x-8}{(x-1)(x-2)}$ into partial fractions.

$$\frac{6x-8}{(x-1)(x-2)} \equiv \frac{A}{x-1}+\frac{B}{x-2}$$
$$\frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

So

$$6x-8 \equiv A(x-2)+B(x-1)$$

Choosing $x=1$ leads to $A=2$.

Choosing $x=2$leads to $B=4$.

Therefore

$$\frac{3x^2-3x-2}{x^2-3x+2} \equiv 3+\frac{2}{x-1}+\frac{4}{x-2}$$

Test yourself...

1. Express $\dfrac{x+16}{(x+1)(x-4)}$ as a partial fraction sum.
2. Express $\dfrac{3x+2}{x(x+1)}$ as a partial fraction sum.
3. Write $\dfrac{3x^2-2x+10}{(x-4)(x+1)^2}$ in terms of partial fractions
4. Write $\dfrac{x+3}{x^3+3x}$ in terms of partial fractions
5. Decompose the expression $\dfrac{2x^3+3x^2-16x+22}{x^2+2x-8}$.

Answers:

1. $\dfrac{4}{x-4}-\dfrac{3}{x+1}$
2. $\dfrac{2}{x}+\dfrac{1}{x+1}$
3. $\dfrac{2}{x-4}+\dfrac{1}{x+1}-\dfrac{3}{(x+1)^2}$
4. $\dfrac{1}{x}+\dfrac{1-x}{x^2+3}$
5. $2x-1+\dfrac{3}{x-2}-\dfrac{1}{x+4}$