Maclaurin Series

Series 4

INU0114/514 (Maths 1)

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Recap: Binomial Series

Recall that some functions can be rewritten as a power series in x. For example:

$$(1+x)^4 \equiv 1+4x+6x^2+4x^3+x^4$$

This is a finite series. It was obtained by multiplying out the brackets, or by using the rules for expanding binomials.

Here is another:

$$(1+2x)^{\frac{1}{2}} \equiv 1+x-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{5}{8}x^4+\ldots$$

This is an infinite series, with no final term on the RHS.

The methods we learned for expanding binomials (e.g. Pascal's triangle, or equivalent) are restricted to functions of the form $(a+x)^n$.

Sometimes we need power series for other functions...

Power series

Colin Maclaurin (1698 — 1746)

Consider the functions

$$f(x) = e^x \quad \text{or} \quad f(x) = \sin x$$

How can we find a power series for these ?
(There are no “brackets” to expand!)

The answer lies with calculus; we can use differentiation to generate the series.

Pioneers in this approach to the analysis of functions include the mathematicians James Gregory, Brook Taylor, Isaac Newton and Colin Maclaurin who lived in the 17th and early 18th centuries.

Deriving a power series

Consider the function $f(x) = e^x$.

FIrst, we assume a power series exists and that we can write it like this:

$$e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots \tag{1}$$

where a_0, a_1, a_2, etc are constants we have to find.

Substitute $x = 0$ into (1) to get:

$$\begin{aligned} e^0 &= a_0 + a_1(0) + a_2(0^2) + a_3(0^3) + \dots \\ \therefore 1 &= a_0 \end{aligned}$$

Now differentiate and evaluate at $x = 0$ again:

$$\begin{aligned} e^x &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots \\ e^0 &= a_1 + 2a_2(0) + 3a_3(0^2) + 4a_4(0^3) + \dots \\ 1 &= a_1 \end{aligned}$$

Repeat: differentiate and set $x = 0$

$$\begin{aligned} e^x &= 2a_2 + 6a_3 x + 12a_4 x^2 + \ldots \\ 1 &= 2a_2 \quad \therefore a_2 = \tfrac{1}{2} \end{aligned}$$

Repeat and we find $a_3 = \frac{1}{6}$ and $a_4 = \frac{1}{24}$.

Substituting these into the original series (1) we see:

$$\boxed{e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \ldots}$$

A power series whiderived for $x = 0$

What does it mean?

We just expressed a function as a power series:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \dots$$

The two sides are identical if we substitute $x = 0$.

To find other function values we must use more terms on the RHS. For example you may know the value of the constant e is approximately 2.718. We can check this with the series:

$$e^1 \approx 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} \approx 2.708$$

Summing the first four terms gives a value close to the actual value of e.

If we use a fifth term the sum increases to 2.717. After six terms we get 2.718.

Just as we saw with binomial series — the more terms we use, the better the approximation of series.

General formula for a Maclaurin series

Any differentiable function $f(x)$ can be expressed as a power series of the form:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \tag{2}$$

where $a_0, a_1, \dots$ are constants that we must find.

We will derive a formula to represent *any* differentiable function as a power series.

Substitute $x = 0$ into equation (2) to get $f(0) = a_0$.

If we differentiate the series in 2 we obtain:

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots \tag{3}$$

Substitute $x = 0$ and we obtain $f'(0) = a_1$.

Differentiating (3) and putting $x = 0$:

$$\begin{aligned} f''(x) &= 2a_2 + (2)3a_3 x + 4(3)a_4 x^2 + \dots \\ f''(0) &= 2a_2 \\ \tfrac{1}{2} f''(0) &= a_2 \end{aligned}$$

Differentiating again and setting $x = 0$ gives:

$$\begin{aligned} f'''(x) &= (2)3a_3 + 4(3)(2)a_4 x + \dots \\ f'''(0) &= (2)(3)a_3 \\ \frac{1}{(2)(3)} f'''(0) &= a_3 \end{aligned}$$

It can be shown that $a_4 = \frac{1}{(2)(3)(4)} f^{iv}(0)$ and that $a_k = \frac{1}{k!} f^k(0)$.

Evaluating the constants a_k at $x = 0$ in the power series (2) leads to this:

$$f(x) = f(0) + xf'(0) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^k(0)}{k!}x^k + \dots \qquad (4)$$

This series is called *Maclaurin series.*

It shows that a power series for a given function can be obtained in straightforward way by repeated differentiation and evaluation at $x = 0$.

Maclaurin series

Find the Maclaurin series for the function $\sin x$.

Let's do this in a systematic way by writing down the function and its derivatives and then evaluating each at $x = 0$.

$$\begin{aligned} f(x) &= \sin x & \therefore f(0) &= 0 \\ f'(x) &= \cos x & \therefore f'(0) &= 1 \\ f''(x) &= -\sin x & \therefore f''(0) &= 0 \\ f'''(x) &= -\cos x & \therefore f'''(0) &= -1 \\ f^{iv}(x) &= \sin x & \therefore f^{iv}(0) &= 0 \end{aligned}$$

The pattern of derivatives will repeat after this, so that $f^{v}(0) = f'(0)$.

Substituting into Maclaurin's series (4) gives:

$$\sin x = 0 + (1)x + (0)\frac{x^2}{2!} + (-1)\frac{x^3}{3!} + (0)\frac{x^4}{4!} + (1)\frac{x^5}{5!} - \ldots$$

Simplify this to get:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Some common series

As a test you should try to obtain the following series for yourself:

$$
\begin{aligned}
e^x &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots \\
\sin x &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots \\
\cos x &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots \\
\tan x &= x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\dots \\
\ln(1+x) &= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots \\
(1+x)^n &= 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots
\end{aligned}
$$

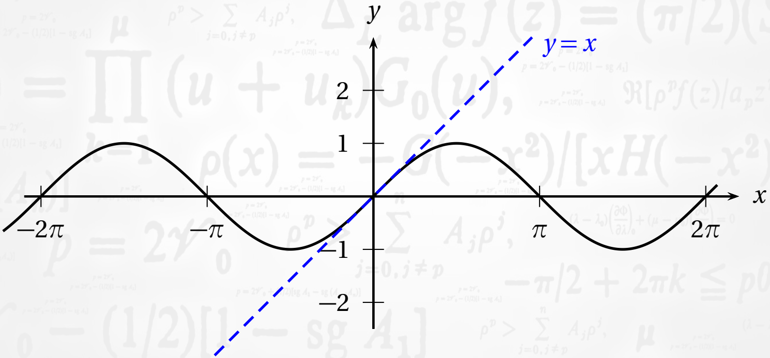
Some functions don't have a Maclaurin series expansion. For example $f(x)=\ln x$ is not defined at $x=0$ and so does not have a Maclaurin series.

The series for $(1+x)^n$ is identical to the *binomial series* seen earlier in the course.

Convergence of Maclaurin Series

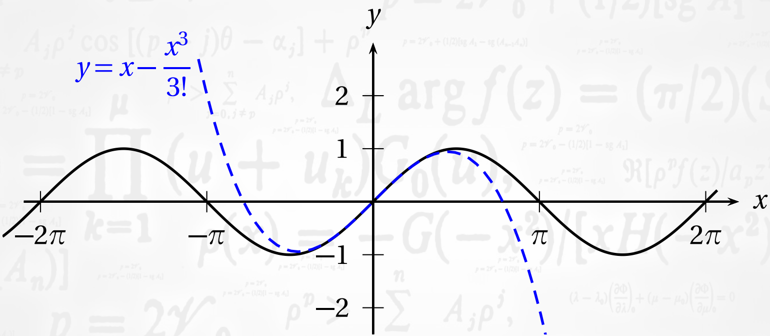
Consider again the series $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

Let's compare how taking more terms on the RHS affects the graph of the series. We'll plot the result against the curve $y = \sin x$.



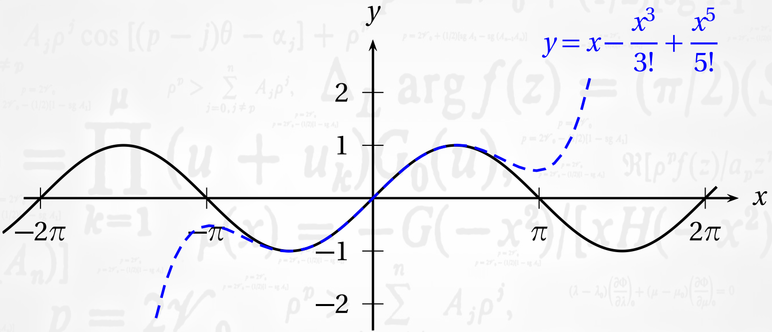
With one term the expansion is $\sin x \approx x$ but it's only valid when x is close to zero. The difference is great at $x = \frac{\pi}{2}$ for example.

Taking the next term in the series:



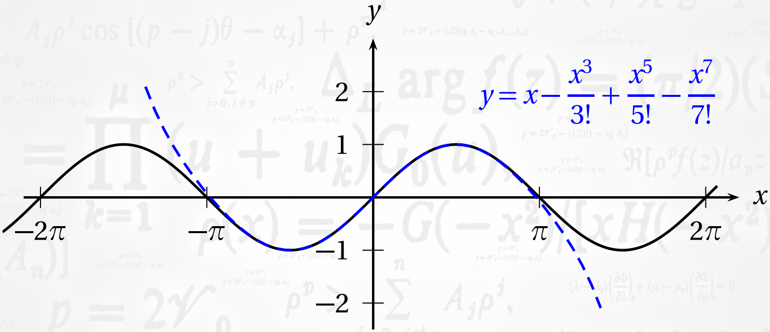
The extra term in the series extends the range of values for which the polynomial matches the sine curve.

Adding another term to the series:



The range of x values for which the two curves are in agreement is extended a little further.

Continuing to use more terms increases the range of x values for which the series is a valid representation of the function $\sin x$.



For *this* series it can be shown that the *radius of convergence* (the values for which the series is valid) can increase without limit — if we use enough terms.

Deriving other series

If we already know a Maclaurin series for a function then we might use it to derive the series of a related function — without doing the differentiation and evaluation at $x = 0$.

For example, the Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots$$

If we need a series for e^{-2x} then we replace each x with $-2x$ in the original:

$$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} \dots$$

Expanding the brackets and simplifying the coefficients:

$$\begin{aligned} e^{-2x} &= 1 - 2x + \frac{4x^2}{2} + \frac{-8x^3}{6} + \frac{16x^4}{24} \dots \\ &= 1 - 2x + 2x^2 - \tfrac{4}{3}x^3 + \tfrac{2}{3}x^4 \dots \end{aligned}$$

Deriving another series

Use the Maclaurin series for e^x and $\cos x$ to obtain the first few terms in the series for

$$f(x) = e^x \cos x$$

This function would rapidly become difficult to differentiate. Instead we'll use the series for each function.

$$e^x \cos x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right)$$

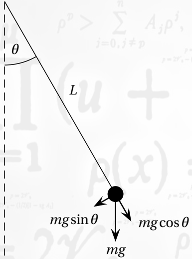
Let's take each term in the first brackets and multiply it with the entire second set of brackets.

We'll only write down the terms which have powers of x of 4 or less.

$$\begin{aligned} e^x \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + x - \frac{x^3}{2} + \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^3}{6} + \dots \\ &= 1 + x - \frac{x^3}{3} - \frac{5x^4}{24} + \dots \end{aligned}$$

An application from physics...

A pendulum of length L at an angle θ to the vertical will swing back and forth.



The equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

Solving this equation would tell us the angle θ at time t.

But this is cannot be solved to find $\theta(t)$ exactly – advanced numerical methods are needed!

If the angle θ is kept small then, because $\sin\theta \approx \theta$, we can write

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

This is much easier to solve! It describes simple harmonic motion and we'll study it in Maths 2 later in the course.

Using terms in a power series to approximate a function can make the maths much easier.

Applications of Maclaurin Series

Evaluating limits

Find the value of $\lim_{x\to 0}\dfrac{\sin x}{x}$.

This function is not defined at $x = 0$.

Using the series for $\sin x$ we can see that:

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\left(\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots}{x}\right)$$
$$= \lim_{x\to 0}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\ldots\right)$$

Taking the limit $x \to 0$ and all but the first term will vanish:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

A difficult integral

Evaluate $\int_0^{0.1} e^{x^2}\,dx$

The indefinite integral of the function e^{x^2} does not exist in terms of elementary functions. Instead we'll use its series.

Starting with

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

we obtain the required series by substituting x^2 for x:

$$e^{x^2} = 1 + (x^2) + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \ldots = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \ldots$$

The original integral can be approximated by:

$$\begin{aligned} \int_0^{0.1} e^{x^2}\,dx &\approx \int_0^{0.1} \left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6}\right) dx \\ &\approx \left[x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42}\right]_0^{0.1} \\ &\approx 0.1003343 \text{ (7 D.P.)} \end{aligned}$$

Binomial and Maclaurin series

- In some mathematical situations that scientists and engineers encounter, it is preferable to replace a function with its power series expansion. The series might simplify a particular piece of maths or make it easier to analyse.
- We have now seen two methods to do this: with a *binomial series* or a *Maclaurin series.*
- Maclaurin series can be used for any function, while a binomial series can only be found for functions of the form $(a+b)^n$.
- Binomial series for $(1+x)^n$ converges when $|x|<1$. It is more complicated to find values for which Maclaurin series converges. Some series converge for $|x|<\infty$ (e.g. $\sin x$). Others do not. Testing for convergence is beyond the scope of this course but if you're interested you should research "radius of convergence" for series.
- A more general version of Maclaurin series, valid for other values of x, is called Taylor series. We don't study that series on this course but you will see it later in your mathematical career.

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Expand $\sqrt{x+4}$ as a Maclaurin's series up to the x^3 term.
2. Find the first three terms in the expansion of e^{-x}.
3. Given that
$$\ln(1+x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \dots$$
find an expansion for $\ln(1-2x)$
4. Find the first three terms of $x^2 e^{-x}$

Answers:

1. $(x+4)^{\frac{1}{2}} = 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3 + \dots$
2. $e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots$ (leaving factorials in is fine!)
3. $\ln(1-2x) = -2x - 2x^2 - \frac{8}{3}x^3 \dots$
4. Multiply the series in (2) by x^2 to get $x^2 e^{-x} = x^2 - x^3 + \frac{1}{2}x^4 - \frac{1}{6}x^5 + \dots$