Binomial Series — Part 3

Series 3

INU0114/514 (Maths 1)

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Objectives

The purpose of this session is to introduce power series expansions and the binomial series in particular.

- Introduction to power series (infinite series)
- Binomial expansions of $(1+x)^n$ and $(a+b)^n$ when n is not a positive integer.
- Convergence of an infinite binomial expansion.
- Applications and examples.

Scientists sometimes need to represent functions by infinite series to simplify a situation or to gain an insight into how functions behave. For example, the exponential function e^x was found to be related to the trigonometric functions $\sin x$ and $\cos x$ by power series.

Binomial series expansions

We saw previously that the expansion of $(a+b)^n$, where n is a positive integer can be obtained using:

$$\begin{aligned} (a+b)^n &= \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + b^n \\ \text{or } (a+b)^n &= a^n + n a^{n-1} b^1 + \frac{n(n-1)}{2!} a^{n-2} b^2 + \ldots + b^n \end{aligned}$$

The expansion of $(1+x)^n$ for positive integer n can be obtained from:

$$\begin{aligned} (1+x)^n &= 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + x^n \\ \text{or } (1+x)^n &= 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 \ldots + x^n \end{aligned}$$

These are valid for when n is a positive integer. In this presentation we will learn how to deal with other values of n.

An infinite series

Consider the series

$$1 + x + x^2 + x^3 + x^4 + \dots$$

This is a geometric series. The common ratio is x and the first term is 1.

Now provided that $-1 < x < 1$ the series will have a sum to infinity:

$$S_\infty = \frac{1}{1-x} \quad , |x| < 1$$

Therefore $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

Or equivalently: $$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

The RHS is only equivalent to the LHS provided the common ratio $|x| < 1$. The value of the RHS converges to the value of the RHS as we take more and more terms.

Extending the binomial series to any n

The previous discussion showed that the expansion

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$$

is an infinite series of terms. Furthermore, it is only valid for certain values of x.

For example, the series diverges when $x = 2$ but converges when $x = \frac{1}{2}$.

For the case $(1+x)^n$, where n is a fraction or a negative value then we can't use expansions based on Pascal's Triangle or based on the coefficients $\binom{n}{k}$ because they only apply to positive integer values of n.

However, we can obtain the binomial expansion of $(1+x)^n$, where n is any real number, using:

$$\boxed{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots}$$

and the expansion is only valid when $-1 < x < 1$ (or equivalently $|x| < 1$).

Binomial expansion (negative n)

Expand $(1+x)^{-3}$ as far as the x^3 term.

(In this example n is a negative integer.)

The expansion begins with the terms:

$$(1+x)^{-3} = 1 + (-3)x + \frac{(-3)(-4)}{2!}x^2 + \frac{(-3)(-4)(-5)}{3!}x^3 + \ldots$$

We can simplify the coefficients to get:

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots$$

This expansion is only valid for $|x| < 1$.

Series convergence

Consider the power series

$$(1+x)^{-3} = 1-3x+6x^2-10x^3+\ldots \qquad ,-1<x<1$$

The domain restriction means the RHS is only true for certain values of x.

For example, substituting $x=9$ into both sides:

$$\begin{aligned} 10^{-3} &= 1-3(9)+6(9)^2-10(9)^3+\ldots \\ \therefore 0.001 &= 1-27+486-7290+\ldots \\ 0.001 &= -6830+\ldots \end{aligned}$$

Clearly this can't be right! The RHS doesn't converge because we chose outside the domain restriction.

Now choose $x=0.1$.

$$\begin{aligned} 1.1^{-3} &= 1-3(0.1)+6(0.1)^2-10(0.1)^3+\ldots \\ \therefore 0.751315 &= 1-0.3+0.06-0.01\ldots \\ 0.751315 &= 0.75+\ldots \end{aligned}$$

This is better; taking more terms on the RHS is causing convergence. The RHS will approach the value of the LHS if we keep adding more terms.

Binomial expansion (fractional n)

Write down the expansion of $\sqrt{1+x}$ up to the term in x^3.

The expansion begins as follows:

$$(1+x)^{\frac{1}{2}} = 1 + \left(\tfrac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \dots$$

Simplify the coefficients:

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + \dots$$

Again, this expansion is only valid for $|x| < 1$.

Convergence of an infinite series

Expand the expression $(1-3x)^{-2}$ to the term in x^3.

Expanding the binomial series in the usual way:

$$\begin{aligned}(1-3x)^{-2} &= 1+(-2)(-3x)+\frac{(-2)(-3)}{2!}(-3x)^2 \\ &\quad +\frac{(-2)(-3)(-4)}{3!}(-3x)^3 \\ &= 1+6x+\frac{(-2)(-3)}{2}\left(9x^2\right) \\ &\quad +\frac{(-2)(-3)(-4)}{6}\left(-27x^3\right) \\ &= 1+6x+27x^2+108x^3\end{aligned}$$

This expansion is valid provided that $|3x|<1$, i.e. provided that $|x|<\frac{1}{3}$, (or $-\frac{1}{3}<x<\frac{1}{3}$).

Convergence (again)

Consider the function $f(x) = \dfrac{1}{1+x}$.

The function can be represented by a binomial expansion

$$f(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$$

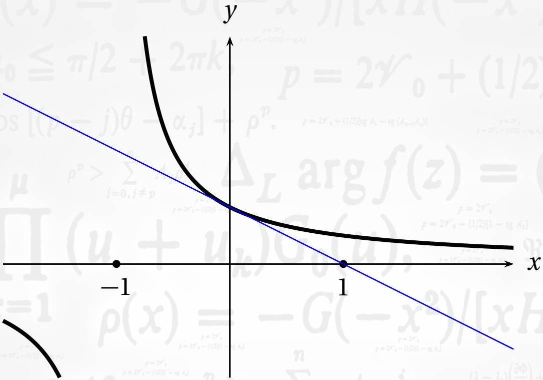
and where $-1 < x < 1$.

We can observe the effects of the expansion using a graph, by taking more terms on the graph of the function.

As an approximation we could use the first two terms of the expansion:

$$f(x) \approx 1 - x$$

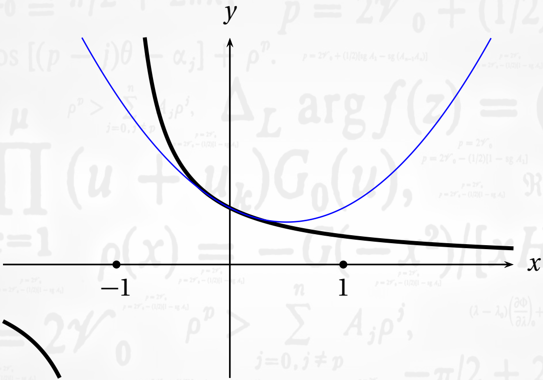
This is the graph of $f(x) = (1+x)^{-1}$ and the expansion $f(x) = 1 - x$.



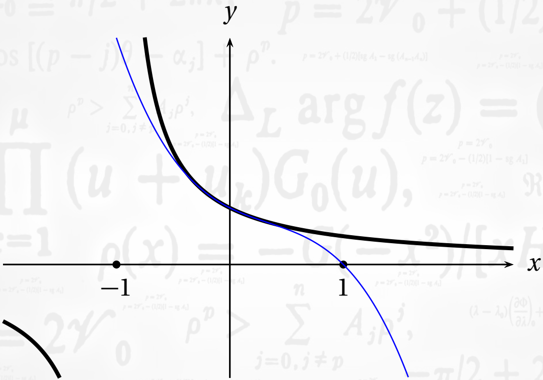
Even though the expansion is valid in the interval $-1 < x < 1$, the line is not a good approximation.

We can get a better approximation to the function by using more terms in the expansion.

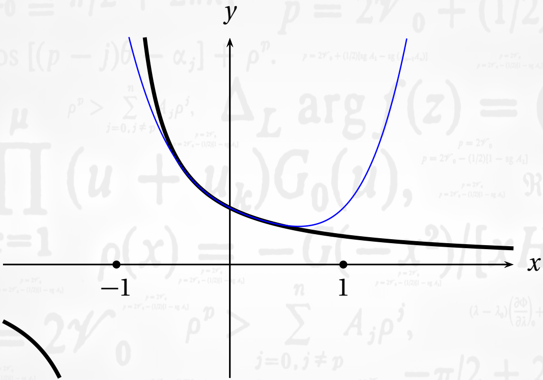
The approximation $f(x) = 1 - x + x^2$



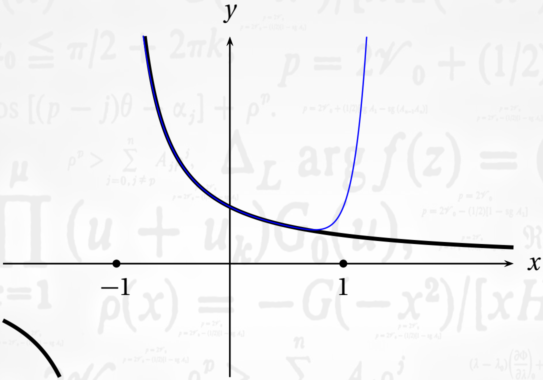
The approximation $f(x) = 1 - x + x^2 - x^3$



The approximation $f(x) = 1 - x + x^2 - x^3 + x^4$



The approximation $f(x) = 1 - x + x^2 - x^3 + x^4 - \ldots + x^{10}$



When 11 terms are included the fit is much better — but only over the interval $-1 < x < 1$.

Outside that interval, the series approximation diverges very quickly!

Uses of binomial series #1

Calculator fail!

Calculate the value of $1 - \dfrac{1}{\sqrt{1 - 1.5 \times 10^{-12}}}$

Attempting find this value on many calculators results in zero. But the answer is not zero.

Binomial expansion provides a better answer.

Expand the first two terms of $(1 - 1.5 \times 10^{-12})^{-\frac{1}{2}}$

$$(1 - 1.5 \times 10^{-12})^{-\frac{1}{2}} \approx 1 + 7.5 \times 10^{-13}$$

Therefore

$$1 - \frac{1}{\sqrt{1 - 1.5 \times 10^{-12}}} \approx 1 - (1 + 7.5 \times 10^{-13}) \approx -7.5 \times 10^{-13}$$

Uses of binomial series #2

Approximating a function with a power series

If x is small, prove that

$$\frac{1+2x}{1+x} \approx 1+x-x^2$$

First rewrite the function on the LHS as a product:

$$\text{LHS} = (1+2x)(1+x)^{-1}$$

We'll expand the term in brackets using the rule for $(1+x)^n$

$$\begin{aligned}\text{LHS} &= (1+2x)\left(1+(-1)x+\frac{(-1)(-2)}{2!}x^2+\dots\right)\\ &= (1+2x)(1-x+x^2\dots)\\ &= 1+2x-x-2x^2+x^2-2x^3\dots\\ &= 1+x-x^2-2x^3\dots\end{aligned}$$

Ignore the terms higher than x^2 to get the approximation:

$$\text{LHS} \equiv \frac{1+2x}{1+x} \approx 1+x-x^2$$

Uses of binomial series #3

Evaluating limits

Evaluate the limit given by $\lim_{x\to 0}\dfrac{(1-2x)^{\frac{3}{2}}-1}{x}$

This function isn't defined at $x=0$.

Replace the binomial part of this with a few terms its expansion:

$$(1-2x)^{\frac{3}{2}} \quad = \quad 1+(\tfrac{3}{2})(-2x)+\frac{(\frac{3}{2})(\frac{1}{2})(-2x)^2}{2!}+\ldots = -3x+\tfrac{3}{2}x^2+\ldots$$

The limit can be expressed like this:

$$\lim_{x\to 0}\frac{(1-2x)^{\frac{3}{2}}}{x} \quad = \quad \lim_{x\to 0}\frac{\left(1-3x+\frac{3}{2}x^2+\ldots\right)-1}{x}$$

$$= \quad \lim_{x\to 0}\frac{-3x+\frac{3}{2}x^2+\ldots}{x} = \lim_{x\to 0}(-3+\tfrac{3}{2}x+\ldots)$$

Let $x\to 0$. Only the first term will remain.

$$\lim_{x\to 0}\frac{(1-2x)^{\frac{3}{2}}-1}{x} = -3$$

General binomial expansion

Show that

$$(8+x)^{\frac{1}{3}} \approx a + bx + cx^2$$

where a, b and c are constants.

We must find a way to express this in terms of $(1+x)^n$.

First — take a factor of 8:

$$(8+x)^{\frac{1}{3}} = \left[8\left(1+\tfrac{1}{8}x\right)\right]^{\frac{1}{3}} = 8^{\frac{1}{3}}(1+\tfrac{1}{8}x)^{\frac{1}{3}} = 2(1+\tfrac{1}{8}x)^{\frac{1}{3}}$$

Now we can continue as usual:

$$\begin{aligned}
(8+x)^{\frac{1}{3}} &= 2(1+\tfrac{1}{8}x)^{\frac{1}{3}} \\
&= 2\left(1+\tfrac{1}{3}(\tfrac{1}{8}x)+\frac{\tfrac{1}{3}(-\tfrac{2}{3})(\tfrac{1}{8}x)^2}{2!}+\ldots\right) \\
&= 2\left(1+\tfrac{1}{24}x-\tfrac{1}{576}x^2+\ldots\right) \\
\therefore (8+x)^{\frac{1}{3}} &= 2+\tfrac{1}{12}x-\tfrac{1}{288}x^2+\ldots
\end{aligned}$$

The domain of the expansion is found by solving $\left|\frac{1}{8}x\right| < 1$:

$$\therefore -8 < x < 8$$

Binomial series coefficients

Find the coefficient of x^2 in the expansion of

$$\sqrt{\frac{1+ax}{4-x}}$$

Give your answer in terms of a.

The expression can be rewritten as $(1+ax)^{\frac{1}{2}}(4-x)^{-\frac{1}{2}}$.

Therefore: $(1+ax)^{\frac{1}{2}} = 1 + \frac{1}{2}ax - \frac{1}{8}a^2x^2 + \dots$

and $(4-x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}(1-\frac{1}{4}x)^{-\frac{1}{2}} = \frac{1}{2}(1+\frac{1}{8}x+\frac{3}{128}x^2+\dots)$

Combining these binomial expansions:

$$(1+ax)^{\frac{1}{2}}(4-x)^{-\frac{1}{2}} \approx \frac{1}{2}\left(1+\frac{1}{2}ax-\frac{1}{8}a^2x^2\right)\left(1+\frac{1}{8}x+\frac{3}{128}x^2\right)$$

Finally, multiply only the terms which give x^2:

$$\frac{1}{2}\left(\frac{3}{128}x^2+\frac{1}{16}ax^2-\frac{1}{8}a^2x^2\right)$$

Taking a common factor, the coefficient of x^2 is

$$\frac{3}{256}+\frac{1}{32}a-\frac{1}{16}a^2$$

Approximating roots

Use a binomial expansion to find the value of $\sqrt{9.18}$ to five decimal places.

We can write the square-root in form which can be expanded like this:

$$\begin{aligned}(9.18)^{\frac{1}{2}} &= (9+0.18)^{\frac{1}{2}} \\ &= [9(1+0.02)]^{\frac{1}{2}} \\ (9.18)^{\frac{1}{2}} &= 3(1+0.02)^{\frac{1}{2}}\end{aligned}$$

Expand using the formula for $(1+x)^n$

$$\begin{aligned} &= 3\left[1+(\tfrac{1}{2})(0.02)+\frac{(\frac{1}{2})(-\frac{1}{2})(0.02)^2}{2!}+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(0.02)^3}{3!}+\ldots\right] \\ &\approx 3(1+0.01-0.00005+0.0000005) \\ &\approx 3+0.03-0.00015+0000015 \\ \therefore \sqrt{9.18} &\approx 3.02985\end{aligned}$$

Summary of binomial series expansions

- Binomial theorem shows us how to expand $(1+x)^n$ or $(a+b)^n$.
- If n is a positive integer we might use either of the following formulae:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}\ldots + \binom{n}{k}x^k + \ldots + x^n$$

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$$

Where $\binom{n}{k}$ are the coefficients of x powers obtained from Pascal's Triangle or the ${}^nC_r$ function of the calculator.

- If n is negative, or not an integer, then we should use the expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}\ldots + \binom{n}{k}x^k + \ldots$$

This is an infinite series but we might just require the first few terms of the expansion (written with ascending powers of x). For an expression of the form $(a+b)^n$ we must factorise first before using this formula.

$$(a+b)^n = \left[a\left(1+\frac{b}{a}\right)\right]^n = a^n\left(1+\frac{b}{a}\right)^n$$

- Infinite series are only valid if they *converge*. The values for which $(1+x)^n$ converge are $|x| < 1$ (meaning $-1 < x < 1$).

Test yourself...

Try the following problems; use binomial expansions up to and including the x^3 term.

1. Expand $(1+3x)^{\frac{1}{3}}$.
2. Expand $(1-\frac{1}{2}x)^{-2}$.
3. For what values of x will the series for $(2+3x)^{-1}$ converge?
4. Use the first two terms of the binomial expansion of $\sqrt{4+x}$ to show that $\sqrt{3.92} \approx \frac{99}{50}$.

Answers:

1. $1+x-x^2+\frac{5}{3}x^3\ldots$
2. $1+x+\frac{3}{4}x^2+\frac{1}{2}x^3\ldots$
3. $-\frac{2}{3}<x<\frac{2}{3}$
4. First: $(4+x)^{\frac{1}{2}} \approx 2+\frac{1}{4}x$. Set $x=-0.08=-\frac{2}{25}$ and $\sqrt{3.92} \approx 2+\frac{1}{4}(-\frac{2}{25})=\frac{99}{50}$.

Linearising a nonlinear function

This is an optional topic — you will not get anything like this on the exam! However, it shows how this topic fits into analysis of functions in physics.

The gravitational acceleration experienced by a particle at the surface of the Earth (mass M) is

$$g = \frac{GM}{R^2}$$

where R is the radius of the Earth and G is the universal constant of gravitation.

What happens to g if we move away from the Earth's surface?

If the particle is moved to a height h above the surface then the corresponding acceleration g_h given by

$$g_h = \frac{GM}{(R+h)^2}$$

We can use a binomial expansion to find a simpler relationship. Remember: G, M and R are constants:

$$\begin{aligned} g_h &= GM(R+h)^{-2} \\ &= GMR^{-2}\left(1+\frac{h}{R}\right)^{-2} \end{aligned}$$

Expand the brackets and take the first term of the series:

$$g_h \approx \frac{GM}{R^2}\left(1-\frac{2h}{R}\right)$$

The gravitational acceleration at height h is given by

$$g_h \approx g\left(1-\frac{2h}{R}\right)$$

We replaced the original nonlinear function for g with a linear version which shows how g varies with h.

We used a binomial series to find how acceleration due to gravity g_h varies with height h above the surface of the Earth. We derived the expression:

$$g_h \approx g\left(1 - \frac{2h}{R}\right)$$

The constants are $g = 9.80665\ \text{m s}^{-2}$ (this is standard gravitational acceleration) and $R = 6\,371\,009$ m (mean radius).

Let's calculate the value of g at the top of Mount Everest (highest mountain on Earth).

Mount Everest (in Nepal) has a height of $h = 8\,848$ m above sea level:

$$g_h \approx 9.80665 \times \left(1 - \frac{2 \times 8848}{6371009}\right) = 9.80665 \times 0.997224 = 9.7794\,\text{m s}^{-2}$$

Not too different to the surface value!

How high above the ground would you have to be to experience $g = 8 \text{ m s}^{-2}$?

Using the approximation:

$$g_h \approx g\left(1 - \frac{2h}{R}\right)$$

Rearrange for h:

$$h \approx \frac{R(g - g_h)}{2g}$$

Substitute $g_h = 8$ (along with $g = 9.80665 \text{ m s}^{-2}$, $R = 6\,371\,009$ m).

We find that

$$h \approx 586\,856.03 \text{ m}$$

The gravitational acceleration falls to 8 m s^{-2} at a height of 587 km.

That's a little higher than the International Space Station usually orbits! Astronauts orbiting the Earth still experience gravity; they are weightless because they are in free fall around the Earth.