Binomial Series — Part 2

Series 3

INU0114/514 (Maths 1)

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Objectives

In this presentation we continue our study of the binomial expansion.

- Review of $(a+b)^n$
- Expressing binomial coefficients in terms of n.
- Expansion of $(1+x)^n$

In each case n is a positive integer. We'll cover other cases in the next presentation!

Binomial theorem

The binomial theorem shows us how to quickly represent a power of $a+b$ as an expanded sum. With Sigma notation it is like this:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

where n is a positive integer and $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ gives the binomial coefficient.

The expansion of $(a+b)^n$ has the following form:

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

Simplifying binomial coefficients

The formula for calculating the coefficients is:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Each coefficient can be simplified to reveal a simple pattern depending on *n*.

$$\binom{n}{0} = \frac{n!}{1!(n-0)!} = \frac{n!}{n!} = 1$$

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n \times (n-1)!}{(n-1)!} = n$$

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n \times (n-1) \times (n-2)!}{2! \times (n-2)!} = \frac{n(n-1)}{2!}$$

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n \times (n-1) \times (n-2) \times (n-3)!}{3! \times (n-3)!} = \frac{n(n-1)(n-2)}{3!}$$

...and so on.

Expansion of $(a+b)^n$ for positive integer n

We've seen that the expansion of $(a+b)^n$ has the following form:

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

where $\binom{n}{k}$ is a function which gives the binomial coefficient.

The coefficients of the binomial expansion formulae can be written in terms of n.

Doing that makes them more convenient to write down and manipulate.

The expansion written above becomes

$$(a+b)^n = a^n + na^{n-1} b^1 + \frac{n(n-1)}{2!} a^{n-2} b^2 + \ldots + b^n$$

Expansion of $(1+x)^n$ for positive integer n

If we let $a = 1$ and replace b with x in the previous binomial expansion formula we get a useful variation:

$$\begin{aligned}(1+x)^n &= \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n \\ &= 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1} + x^n\end{aligned}$$

And this can also be written in the following form:

$$\boxed{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots + x^n}$$

A binomial expansion

Expand $(1+x)^4$.

We must calculate the following coefficients:

$$\begin{aligned}(1+x)^4 &= 1+4x+\frac{(4)(3)}{2!}x^2+\frac{(4)(3)(2)}{3!}x^3+\frac{(4)(3)(2)(1)}{4!}x^4\\ &= 1+4x+\frac{12}{2}x^2+\frac{24}{6}x^3+\frac{24}{24}x^4\end{aligned}$$

Therefore

$$(1+x)^4 = 1+4x+6x^2+4x^3+x^4$$

Binomial expansion

Expand $(1-2x)^5$.

We must calculate the following coefficients:

$$\begin{aligned}(1-2x)^5 &= 1+5(-2x)+\frac{(5)(4)}{2!}(-2x)^2+\frac{(5)(4)(3)}{3!}(-2x)^3\\ &\quad+\frac{(5)(4)(3)(2)}{4!}(-2x)^4+\frac{(5)(4)(3)(2)(1)}{5!}(-2x)^5\\ &= 1-10x+\frac{20}{2}(4x^2)+\frac{60}{6}(-8x^3)+\frac{120}{24}(16x^4)+\frac{120}{120}(-32x^5)\end{aligned}$$

Therefore

$$(1-2x)^5 = 1-10x+40x^2-80x^3+80x^4-32x^5$$

Binomial expansion

Write down the expansion of $(1-\frac{1}{2}x)^3$.

Using the expansion formula for $(1+x)^n$

$$\begin{aligned}\left(1-\tfrac{1}{2}x\right)^3 &= 1+(3)(-\tfrac{1}{2})x+\frac{3(2)}{2!}(-\tfrac{1}{2}x)^2+\frac{3(2)(1)}{3!}(-\tfrac{1}{2}x)^3 \\ &= 1-\tfrac{3}{2}x+\tfrac{3}{4}x^2-\tfrac{1}{8}x^3\end{aligned}$$

Binomial expansion using $(1+x)^n$

Expand $(2+x)^8$ up to and including the x^3 term.

Rather than use $(a+b)^n$, let's factorise and use the other expansion:

$$\begin{aligned}(2+x)^8 &= \left[2(1+\tfrac{1}{2}x)\right]^8 \\ &= 2^8\left(1+\tfrac{1}{2}x\right)^8 \\ &= 256\left(1+\tfrac{1}{2}x\right)^8\end{aligned}$$

Now expand the brackets using the rule for $(1+x)^n$:

$$\begin{aligned} &= 256\left[1+8(\tfrac{1}{2}x)+\frac{8(7)}{2!}(\tfrac{1}{2}x)^2+\frac{8(7)(6)}{2!}(\tfrac{1}{2}x)^3\ldots\right] \\ &= 256\left(1+4x+7x^2+7x^3+\ldots\right)\end{aligned}$$

Therefore $(2+x)^8 = 256+1024x+1792x^2+1792x^3+\ldots$

Summary

To expand $(1+x)^n$ we can use

$$
\begin{aligned}
(1+x)^n &= 1+\binom{n}{1}x+\binom{n}{2}x^2+\ldots+x^n \\
\text{or } (1+x)^n &= 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}\ldots+x^n
\end{aligned}
$$

These formulae only apply when n is a positive integer.

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Expand $(1-x)^4$
2. Given $(1+x^2)^3$
3. Given that $(1+4x)^6 = a + bx + cx^2 + \ldots$, find a, b and c.
4. Use (3) to find the approximate value of $(1.04)^6$.

Answers:

1. $1-4x+6x^2-4x^3+x^4$.
2. $1+3x^2+3x^4+x^6$.
3. $1+24x+240x^2$. ($a=1$, $b=24$, $c=240$)
4. Put $x=0.01$ so that $(1.04)^6 = (1+0.04)^6 = (1+4(0.01))^6 \approx 1+4(0.01)+240(0.01)^2 \approx 1.064$.