Binomial Series — Part 1

Series 3

INU0114/514 (Maths 1)

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Objectives

The purpose of this session is to introduce power series expansions and the binomial series in particular.

- The factorial function.
- Pascal's Triangle.
- Binomial expansions of $(1+x)^n$ and $(a+b)^n$ when n is a positive integer.
- Applications and examples.

Binomial theorem

The binomial theorem shows us how to quickly represent a power of $a+b$ as an expanded sum. Using Sigma notation it is like this:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

where n is a positive integer.

Using the binomial theorem we can prove that

$$(x+y)^4 \equiv x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

without the working through the tedious multiplication of $(x+y)(x+y)(x+y)(x+y)$.

We'll build our understanding of the binomial theorem with factorials and Pascal's triangle.

Expanding binomial terms

Binomial terms

A binomial is the sum (or difference) of two terms. For example, $a+b$, $3x-5$, $1+x^2$ are binomials.

We often need to expand a binomial to some power.

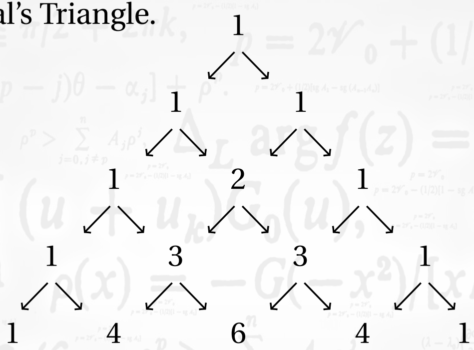
Consider the following expansions:

$$\begin{aligned}(1+x)^0 &= 1\\(1+x)^1 &= 1+x\\(1+x)^2 &= 1+2x+x^2\\(1+x)^3 &= 1+3x+3x^2+x^3\\(1+x)^4 &= 1+4x+6x^2+4x^3+x^4\end{aligned}$$

The coefficients in each row follow a pattern. They are rows in Pascal's Triangle.

Pascal's Triangle

Here is Pascal's Triangle.



The next row of the triangle is obtained by adding elements from row above.

The *coefficients* of the terms in a binomial expansion can be obtained from Pascal's Triangle.

Using Pascal's Triangle

Use the next row of the triangle to expand the binomial expression

$$(1+x)^5$$

The next row of the triangle contains 1, 5, 10, 10, 5 and 1.

The powers of x increase from x^0 to x^5:

$$(1+x)^5 \equiv 1(x^0)+5(x^1)+10(x^2)+10(x^3)+5(x^4)+1(x^5)$$

Simplifying this gives:

$$(1+x)^5 \equiv 1+5x+10x^2+10x^3+5x^4+x^5$$

The binomial $(a+b)^n$

Let's expand the expression $(a+b)^n$ where a is a constant.

$$
\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a+b \\
(a+b)^2 &= a^2+2ab+b^2 \\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4
\end{aligned}
$$

The coefficients are from Pascal's Triangle. Other patterns too:

- The sum of the powers of a and b in each term is equal to n.
- The powers of a start at the value n and decrease as we go from term to term.
- The powers of b start at the value 0 and increase as we go from term to term.
- The are $n+1$ terms in the expansion.

A binomial expansion

Use Pascal's Triangle to expand $(2-x)^4$.

Using all the patterns to expand the expression:

$$\begin{aligned}(2-x)^4 &= 1\left(2^4\right)(-x)^0 + 4\left(2^3\right)(-x)^1 + 6\left(2^2\right)(-x)^2 \\ &\quad +4\left(2^1\right)(-x)^3 + 1\left(2^0\right)(-x)^4 \\ &= 16 + 32(-x) + 24\left(x^2\right) + 8\left(-x^3\right) + x^4 \\ \therefore (2-x)^4 &= 16 - 32x + 24x^2 - 8x^3 + x^4\end{aligned}$$

Factorial function

The factorial function is only defined for non-negative integers.

The factorial of n is denoted by $n!$ and is calculated as follows:

$$\boxed{n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1} \quad (1)$$

So, for example $5! = 120$ (since $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$).

In the same way you should be able to show that $7! = 5040$.

It is not obvious under this definition what the value of $1!$ or $0!$ should be.

By convention the following definitions are used:

$$1! = 1 \qquad \text{and} \qquad 0! = 1$$

Most scientific calculators are able to calculate the factorial of a number; it is usually accessed with a button like $\boxed{n!}$ or $\boxed{x!}$.

Calculating the coefficients

Pascal's triangle isn't efficient for large values of n. Suppose we needed the 100th row?

The k^{th} coefficient in row n of Pascal's triangle can be can be calculated using the formula

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

This formula is related to Pascal's triangle like this:

$n=0$	1						
$n=1$	1	1					
$n=2$	1	2	1				
$n=3$	1	3	3	1			
$n=4$	1	4	6	4	1		
$n=5$	1	5	10	10	5	1	
$n=6$	1	6	15	20	15	6	1
$k \longrightarrow$	0	1	2	3	4	5	6

Calculating coefficients

Calculate $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

This is the 4th row of Pascal's Triangle and the 3rd coefficient.

$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!1!} = \frac{24}{6 \times 1} = 4$$

It usually isn't necessary to use this formula.

We use binomial coefficients so often that most calculators have the formula built into them; look for a button marked with $\boxed{{}_nC_r}$ (or equivalent) to access the function.

For example: to evaluate $\begin{pmatrix} 10 \\ 5 \end{pmatrix}$

Input to the calculator $10 \rightarrow \boxed{{}_nC_r} \rightarrow 5$ to get the answer 252.

Expansion of $(a+b)^n$

The results obtained in the previous sections and examples suggest that the expansion of $(a+b)^n$ has the following form:

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

where $\binom{n}{k}$ is a function which gives the binomial coefficient.

For example, to expand $(2+x)^3$ we could write:

$$(2+x)^3 = \binom{3}{0} 2^3 x^0 + \binom{3}{1} 2^2 x^1 + \binom{3}{2} 2^1 x^2 + \binom{3}{3} 2^0 x^3$$

Use Pascal's triangle or the $\boxed{{}_nC_r}$ button to simplify:

$$\begin{aligned} (2+x)^3 &= (1)(8)(1) + (3)(4)(x) + (3)(2)(x^2) + (1)(1)(x^3) \\ &= 8 + 12x + 6x^2 + x^3 \end{aligned}$$

Binomial expansion

Write down the full expansion of $\left(4+\frac{3}{2}x\right)^4$.

The expansion we require is:

$$\begin{aligned}\left(4+\tfrac{3}{2}x\right)^4 &= \binom{4}{0}(4)^4(\tfrac{3}{2}x)^0+\binom{4}{1}(4)^3(\tfrac{3}{2}x)^1+\binom{4}{2}(4)^2(\tfrac{3}{2}x)^2\\ &\quad+\binom{4}{3}(4)^1(\tfrac{3}{2}x)^3+\binom{4}{4}(4)^0(\tfrac{3}{2}x)^4\end{aligned}$$

Simplifying the coefficients and powers:

$$\begin{aligned}\left(4+\tfrac{3}{2}x\right)^4 &= (1)(256)(1)+(4)(64)(\tfrac{3}{2}x)+(6)(16)(\tfrac{9}{4}x^2)\\ &\quad+(4)(4)(\tfrac{27}{8}x^3)+(1)(1)(\tfrac{81}{16}x^4)\\ &= 256+384x+216x^2+54x^3+\tfrac{81}{16}x^4\end{aligned}$$

Evaluate binomial coefficients

Given that $(1+x)^{50} = a + bx + cx^2 + dx^3 + \dots$
Find the values of a, b, c and d.

We must calculate the following coefficients:

$$(1+x)^{50} = \binom{50}{0}x^0 + \binom{50}{1}x^1 + \binom{50}{2}x^2 + \binom{50}{3}x^3 + \dots$$

Therefore

$$(1+x)^{50} = 1 + 50x + 1225x^2 + 19600x^3 + \dots$$

Application: binomial approximations

The gravitational acceleration experienced by a particle at the surface of the Earth (mass M) is

$$g = \frac{GM}{R^2}$$

where R is the radius of the Earth and G is the universal constant of gravitation.

What happens to g if we move away from the Earth's surface? It can be shown that the gravitational acceleration at height h is given by

$$g_h \approx g\left(1 - \frac{2h}{R}\right)$$

Using binomial series we replace the original nonlinear function for g with a linear version which shows how g varies with h.

Before we learn how to deal with expressions like this - let's start with simple examples containing only numbers!

Approximating powers

Write down the expansion of $(2+x)^4$ up to and including the x^2 term. Then use the expansion to evaluate:
(a) $(2.01)^4$
(b) $(1.99)^4$

Expanding the brackets using the binomial theorem gives:

$$(2+x)^4 \approx 16+32x+24x^2$$

(a) Substitute $x = 0.01$

$$\begin{aligned} (2.01)^4 &= (2+0.01)^4 \\ &\approx 16+32(0.01)+24(0.01)^2 \\ &\approx 16+0.32+0.0024 \\ &\approx 16.3224 \end{aligned}$$

(b) Substitute $x = -0.01$

$$
\begin{aligned}
(1.99)^4 &= (2 - 0.01)^4 \\
&\approx 16 + 32(-0.01) + 24(-0.01)^2 \\
&\approx 16 - 0.32 + 0.0024 \\
&\approx 15.6824
\end{aligned}
$$

We'll examine binomial approximations in more detail soon.

Summary

The binomial series is

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

It is useful for expanding binomial terms for higher powers of n.

The expanded form of this formula is

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + b^n$$

These formulae only apply when n is a positive integer.

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Write down the first five rows of Pascal's triangle in less than 1 minute!
2. Evaluate $\binom{7}{3}$ and $\binom{30}{3}$.
3. Expand $(3+2x)^4$
4. Expand $(1-\frac{1}{2}x)^3$

Answers:

1. See the example given earlier in the notes!
2. 35 and 4060.
3. $81+216x+216x^2+96x^3+16x^4$.
4. $1-\frac{3}{2}x+\frac{3}{4}x^2-\frac{1}{8}x^3$.