Geometric Series

Series 2

INU0114/514 (Maths 1)

Dr Adrian Jannetta MIMA CMath FRAS

Objectives

The purpose of this session is to understand how to deal with geometric series.

- Definition of a geometric series
- Find the n^{th} term
- Find the sum of n terms.
- Convergence and divergence.
- Calculating the sum to infinity of a convergent geometric series

We will practice using Sigma notation in several of the examples.

Achilles and the Tortoise

Zeno of Elea (c490 — c435 BC) proposed that in a footrace between Achilles and a tortoise, Achilles would always lose the race if the tortoise was given a headstart.

Imagine that the tortoise begins the race 10 metres in front of Achilles. By the time Achilles runs 10 metres then the tortoise will have moved further.

Achilles can never catch the tortoise because he is always running to pass the distance the tortoise has already covered!

How do we explain and resolve this paradox?

Geometric series

Geometric series: definition

An geometric series is a series where each term differs from the previous term by a constant multiple. The fixed multiple by which adjacent terms in the series differ is called the *common ratio*. It is denoted by r.

Geometric series are sometimes called *geometric progressions*.

For example, in the geometric series

$$1+2+4+8\ldots+256$$

the common ratio is $r=2$.

The common ratio can be found simply by dividing any term in the series by the one before it.

For the series which begins

$$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\ldots$$

the common ratio is $r=-\frac{1}{3}$.

For the general geometric series given by

$$a + ar + ar^2 + ar^3 + ar^4 + \dots$$

The terms are related by the common ratio r:

1st	a
2nd	ar
3rd	$(ar)r = ar^2$
4th	$(ar^2)r = ar^3$
5th	$(ar^3)r = ar^4$

The 6th term can be expressed as ar^5 and so on.

We usually denote the first term by a and so to calculate the nth term a_n we use the formula:

$$\boxed{n\text{th term, } a_n = ar^{n-1}} \quad (1)$$

Calculating the n^{th} term in a geometric series

Write down the 7th and 10th terms of the geometric series which begins as follows:

$$12 + 6 + 3 + 1.5 + \dots$$

The common ratio in this case is $r = \frac{1}{2}$. The first term is $a = 12$.

The 7th term is a_7 so

$$a_7 = ar^6 = 12(\tfrac{1}{2})^6 = 12(\tfrac{1}{64}) = \tfrac{3}{16}$$

The 10th term is a_{10}:

$$a_{10} = ar^9 = 12(\tfrac{1}{2})^9 = 12\left(\tfrac{1}{512}\right) = \tfrac{3}{128}$$

Sum of n terms of a geometric series

The sum of a geometric series with n terms has the following form:

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} \tag{2}$$

If we multiply both sides by r we obtain

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + ar^n \tag{3}$$

Subtract equation 2 from equation 3 and we see that most of the terms cancel out; only two remain on the RHS.

$$\begin{aligned} S_n - rS_n &= a - ar^n \\ S_n(1-r) &= a(1-r^n) \end{aligned}$$

Dividing both sides by $1-r$ gives a formula for the sum of the first n terms of a geometric series:

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}} \tag{4}$$

Sum of a geometric series

A geometric series begins with the terms:

$$3-6+12-24+\ldots$$

Find the sum of the first 10 terms of the series.

In this case the first term is $a=3$ and the common ratio is $r=-2$.

Using the geometric sum formula $S_n=\dfrac{a(1-r^n)}{1-r}$:

$$\begin{aligned} S_{10} &= \frac{3(1-(-2)^{10})}{1-(-2)} \\ &= \frac{3(1-1024)}{3} \\ S_{10} &= -1023 \end{aligned}$$

Geometric series sum: Sigma notation

Evaluate $\sum_{n=2}^{8} 5^n$

The common ratio is $r = 5$.

The first term of the series (using $n = 2$) is $a = 25$.

The number of terms in the series is 7.

Using the geometric sum formula $S_n = \frac{a(1-r^n)}{1-r}$:

$$\begin{aligned} S_7 &= \frac{25(1-5^7)}{1-5} \\ &= \frac{25(-78124)}{-4} \\ S_7 &= 488\,275 \end{aligned}$$

How many ancestors do you have?

Assuming that one generation corresponds to 25 years, find the number of ancestors — parents, grandparents, etc — a person might have over an 800 year period.

The number of generations is $\frac{800}{25} = 32$ but that includes the person. There are 31 generations of ancestors!

The total we need is

$$S_{32} = 2 + 4 + 8 + 16 + ... + 2^{31} = \sum_{n=1}^{31} 2^n$$

Using the formula

$$S_{32} = \frac{2(1-2^{31})}{1-2} = 2(2^{31} - 1) = 4\,294\,967\,294$$

That's about 4.3 billion ancestors.

Geometric series problem

The third and sixth terms of a geometric series are 12 and −96 respectively.
(a) Find the common ratio of the series
(b) Find the 8^{th} term of the series.
(c) Evaluate the sum of the first 10 terms or the series.

(a) Using the expression for the n^{th} term we can write

$$ar^2 = 12 \quad \text{and } ar^5 = -96$$

Dividing the first equation by the second:

$$\begin{aligned} \frac{ar^5}{ar^2} &= \frac{-96}{12} \\ r^3 &= -8 \\ r &= -2 \end{aligned}$$

(b) Since $r = -2$ the 3^{rd} term can be written as

$$\begin{aligned} 12 &= a(-2)^2 = 4a \\ \therefore a &= 3 \end{aligned}$$

The 8^{th} term can be obtained from

$$a_8 = 3(-2)^7 = 3 \times (-128) = -384$$

(c) For the sum of the first 10 terms of the series:

$$\begin{aligned} S_{10} &= \frac{3\left(1-(-2)^{10}\right)}{1-(-2)} \\ &= \frac{3(1-1024)}{3} \\ &= -1023 \end{aligned}$$

Number of terms less than a given value

The second and fifth terms of a geometric series are $\frac{1}{2}$ and 32 respectively.
(a) Find the first term and common ratio.
(b) Find the number of terms of the series that are less than 10 000.

(a) We are given $ar = \frac{1}{2}$ and $ar^4 = 32$.

Divide the equations:

$$\frac{ar^4}{ar} = \frac{32}{\frac{1}{2}} \quad \Rightarrow r^3 = 64 \quad \therefore r = 4$$

Substitute into the second term:

$$4a = \frac{1}{2} \qquad \therefore a = \frac{1}{8}$$

(b) We must find the n^{th} term so that $a_n < 10\,000$. In other words, we solve:

$$\begin{aligned} ar^{n-1} &< 10\,000 \\ \tfrac{1}{8}(4^{n-1}) &< 10\,000 \\ 4^{n-1} &< 80\,000 \end{aligned}$$

We can solve this using logarithms:

$$\begin{aligned} \log 4^{n-1} &< \log 80\,000 \\ (n-1)\log 4 &< \log 80\,000 \end{aligned}$$

Rearrange to make n the subject. We divide both sides by $\log 4$ (which is positive, so we don't need to reverse the inequality!)

$$n-1 < \frac{\log 80\,000}{\log 4} \quad \Rightarrow n < \frac{\log 80\,000}{\log 4} + 1$$

Evaluating this expression gives $n < 9.143856$ (to 6 D.P.), so $n = 9$.

The 9th term is less than 10 000 (meaning the 10th term would be greater).

Convergence of a geometric series

Consider the following series which begins:

$$S = 1 + 2 + 4 + 8 + 16 + 32 + \dots$$

As we add more terms to each of these the value of S will get larger. The common ratio is $r = 2$ and as $n \to \infty$ so it's obvious that $S \to \infty$ also.

Here is another geometric series:

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \dots$$

The common ratio for this series is $\frac{1}{2}$.

Although the value of S increases each time a term is added, the value of S increases more slowly with each term.

a_n	1	2	3	4	5	...	20
S_n	1	1.5	1.75	1.875	1.9375	...	1.999981

The sum of terms seems to be approaching a limit.

As more terms are added the total appears to be approaching a limiting value of 2. How can we prove, in this case, that $S = 2$ as $n \to \infty$?

Sum to infinity

Consider the formula for the sum of a geometric series:

$$S_n = \frac{a(1-r^n)}{1-r}$$

The growth of the sum of terms is controlled by the term r^n.

If r is numerically less than 1 (meaning $|r| < 1$ or $-1 < r < 1$) then

$$\lim_{n\to\infty} [r^n] = 0$$

In the case where $r > 1$ then

$$\lim_{n\to\infty} [r^n] = \infty$$

In the case where $r = 1$ we cannot immediately say whether r^n will converge or diverge; more investigation is required.

If the geometric series has a finite sum, (i.e. when $-1 < r < 1$) then we can calculate the "sum to infinity" of the series. This is the limiting value of the sum as more and more terms are added.

If we let $n \to \infty$ in the formula for S_n we see it simplifies to:

$$\boxed{S_\infty = \frac{a}{1-r} \qquad \text{provided } |r| < 1}$$

Sum to infinity of a geometric series

For the geometric series:

$$S = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} \ldots$$

Calculate the sum of the terms in this infinite series.

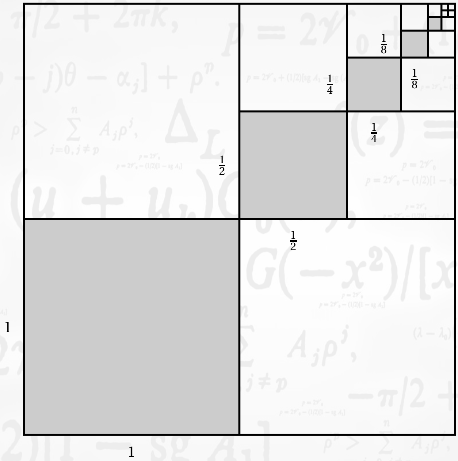
The common ration is $r = \frac{1}{2}$ (which ensures convergence, because it is within $-1 < r < 1$).

The first term is $a = 1$.

The sum to infinity is calculated from:

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{1}{1-\frac{1}{2}} \\ S_\infty &= 2 \end{aligned}$$

A graphical depiction of the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \dots$



An infinite sum in Sigma notation

Evaluate $\sum_{n=1}^{\infty}(-\frac{1}{3})^n$

The first term is $a = -\frac{1}{3}$. The common ratio is $-\frac{1}{3}$.

The sum to infinity is

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{-\frac{1}{3}}{1-(-\frac{1}{3})} \\ &= \frac{-\frac{1}{3}}{\frac{4}{3}} \\ S_\infty &= -\frac{1}{4} \end{aligned}$$

Summary

A geometric series has a first term a and common ratio r:

$$a + ar + ar^2 + ar^3 + \ldots + ar^k + \ldots$$

The n^{th} term of the series is ar^{n-1}.

The sum of the first n terms is given by

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

For an infinite series, the sum will converge to

$$S_\infty = \frac{a}{1 - r}$$

only if $-1 < r < 1$.

Achilles and the Tortoise revisited

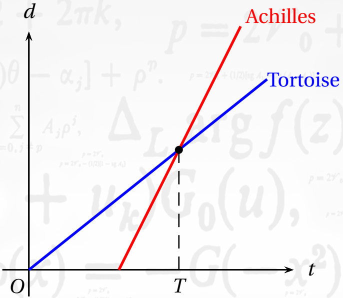
The paradox is rigged to allow the tortoise to win! The paradox is really saying:

During the time before Achilles catches the tortoise — how many instants can it be divided into?

Zeno's paradox is really only about the part of the race where the tortoise is winning — and about how that finite time period can be divided up infinitely. Achilles will pass the tortoise after that.

Achilles v Tortoise

A distance time graph further illustrates the 'paradox'.



The tortoise begins the race at $t = 0$ and Achilles begins running (more quickly) sometime later. Achilles passes the tortoise at time T. According to Zeno the interval between 0 and T can be divided into an infinite number of instants. We've seen how an infinite series can have a finite sum so the paradox is resolved.

Also...in reality time might be quantised and the situation isn't physically possible anyway. Quantum physicists talk of the *Planck Time* (around 10^{-43} seconds) as being the smallest possible unit of time.

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Is this a geometric series: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$?
2. The 2nd and 5th terms of a geometric series are 1 and 27. Find the 10th term.
3. Find the sum of the first 10 terms of the previous series.
4. The sum to infinity of a geometric series is 8 and the common ratio is $-\frac{1}{4}$. Find the first term of the series.

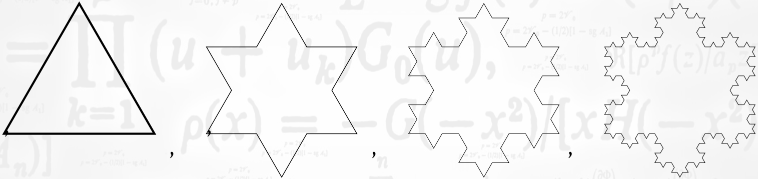
Answers:

1. No; there isn't a common ratio r between terms.
2. $r=3$, $a=\frac{1}{3}$. The 10th term is $ar^9=\frac{1}{3}(3^9)=6561$.
3. $S_{10}=\frac{29524}{3}=9841\frac{1}{3}$.
4. $a=10$.

The Koch Snowflake

The Koch Snowflake is a *fractal* formed after starting with an equalateral triangle and then recursively removing the middle-third of each line and adding adding two line segments to make another equalateral triangle.

The first few iterations of this process are shown in the pictures below.



The area and perimeter of the fractal are the sum of a geometric series.

It can be be shown that, as the number iterations becomes infinite:

- The area of the eventual snowflake is $\frac{8}{5}$ greater than the first triangle.
- The perimeter becomes infinite.

An infinitely long perimeter enclosing a finite area. Can you prove it?