Arithmetic Series

Series 1

INU0114/514 (Maths 1)

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Objectives

The purpose of this session is to understand how to deal with arithmetic series.

- Definition of an arithmetic series
- Finding the n^{th} term of an arithmetic series
- Find the sum of terms in an arithmetic series.

We will practice using Sigma notation in several of the examples.

Arithmetic series

Arithmetic series: definition

An arithmetic series is a series where each term differs from the previous term by a fixed amount. The amount by which the terms in an arithmetic series differ is called the *common difference*.

Arithmetic series are sometimes known as *arithmetic progressions*.

In the arithmetic series

$$3+7+11+15+\ldots+31$$

the common difference is 4.

Given the arithmetic series

$$100+97+94+91+88+85$$

we see that the common difference is -3.

The n^{th} term of an arithmetic series

For the general arithmetic series given by

$$a_1 + a_2 + a_3 + a_4 + a_5 + \ldots + a_k + \ldots$$

and where the common difference between each term is d then we see the following patterns among the terms of the series:

$$\begin{array}{rl} \text{1st} & a_1 \\ \text{2nd} & a_2 = a_1 + d \\ \text{3rd} & a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d \\ \text{4th} & a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d \\ \text{5th} & a_5 = a_4 + d = (a_1 + 3d) + d = a_1 + 4d \end{array}$$

...and so on.

To calculate the nth term in the series the rule for doing so is:

$$\boxed{n^{th} \text{ term: } a_n = a + (n-1)d} \tag{1}$$

where a (i.e. $a = a_1$) is the first term of the series and d is the common difference.

Calculating particular terms in a series

Write down the 10th, 50th and 243rd term of the arithmetic series which begins as follows:

$$2+6+10+14+\dots$$

In this arithmetic series we see that the first term is $a=2$ and the common difference is $d=4$.

The n^{th} term of the series is given by:

$$\begin{aligned} a_n &= a+(n-1)d \\ &= 2+4(n-1) \\ a_n &= 4n-2 \end{aligned}$$

The required terms are:

The 10th term: $a_{10}=4(10)-2=38$

The 50th term: $a_{50}=4(50)-2=198$

The 243rd term: $a_{243}=4(243)-2=970$

Sum of a finite arithmetic series

Finite arithmetic series end after n terms so we can find the sum of the terms.

Consider the sum S for the series

$$S = 1+2+3+4+5+\ldots+99+100$$

Writing the series in reverse underneath:

$$\begin{array}{ccccccccccc} S & = & 1 & + & 2 & +3 & + & \ldots & +99 & + & 100 \\ S & = & 100 & + & 99 & +98 & + & \ldots & +2 & + & 1 \end{array}$$

Do you see the pattern? Each column adds to 101. Adding the two rows together:

$$2S = 101+101+101+\ldots+101+101$$

How many terms are there on the RHS? There are 100. We can simplify the RHS to get:

$$2S = 100 \times 101$$

and the sum of the series is found by dividing both sides by 2:

$$S = \frac{100 \times 101}{2} = 5050$$

Summing an arithmetic series: general case

For an arithmetic series of n terms, where the first term is a, the last term is l and the common difference is d.

The sum of the series is

$$S = a + (a+d) + (a+2d) + (a+3d) + \ldots + (l-2d) + (l-d) + l$$

Writing the sequence in reverse underneath:

$$\begin{array}{ccccccccccccc} S & = & a & + & (a+d) & + & (a+2d) & + & (a+3d) & \ldots & + & l-d & + & l \\ S & = & l & + & (l-d) & + & (l-2d) & + & (l-3d) & \ldots & + & (a+d) & + & a \end{array}$$

Adding these together gives:

$$2S = (a+l) + (a+l) + (a+l) + \ldots + (a+l) + (a+l)$$

There are n identical terms on the RHS so we can simplify this to:

$$2S = n(a+l)$$

Dividing both sides by 2:

$$\boxed{S = \frac{n}{2}(a+l)} \qquad (2)$$

This formula gives the sum of an arithmetic series provided we know the number of terms and the first and last term.

Since the last term is the nth term we know, from equation (1), that $l = a + (n-1)d$ and by substituting into the above equation we obtain:

$$S = \frac{n}{2}[a + a + (n-1)d]$$

So

$$\boxed{S = \frac{n}{2}[2a + (n-1)d]} \quad (3)$$

This is the sum of the series expressed in terms of the number of terms n, the first term a and the common difference d.

Sum of an arithmetic series

An arithmetic series begins

$$15+11+7+3+\ldots$$

Find the sum of the first 25 terms of the series.

The description gives us:

$$a=15, \qquad d=-4, \qquad n=25$$

Using equation (3), the sum is found like this:

$$\begin{aligned} S &= \frac{25}{2}\left[(2\times 15)+(25-1)\times(-4)\right] \\ &= 12.5(30-(24\times 4)) \\ &= 12.5\times(-66) \\ S &= -825 \end{aligned}$$

Arithmetic series with Sigma notation

Evaluate $\sum_{k=1}^{30}(2k+1)$.

We use equation (2) to evaluate the series sum.

There are 30 terms in this series (so $n=30$).

The first term (with $k=1$) is $a=3$.

The last term (with $k=30$) is $l=61$.

The common difference is $d=2$ (since the coefficient of k is 2 in the sum).

Substituting these values into equation (2) we obtain:

$$S=\frac{30}{2}(3+61)=15\times 64=960$$

Arithmetic series problem

The 2nd term of an arithmetic series is 26 and the 5th term is 41.
(a) Find the common difference of the series
(b) Find the 12th term of the series
(c) Find the sum of the first 15 terms.

(a) The 2nd term: $a+d=26$. The 5th term: $a+4d=41$.

Solve these *simultaneous equations* to get $d=5$ (and $a=21$)

(b) The 12th term: $a_{12}=a+11d=21+11(5)=76$.

(c) The sum of the first 15 terms:

$$S_{15}=\frac{15}{2}(2(21)+14(5))=\frac{15}{2}(42+70)=840$$

Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Find the 20th term of an arithmetic series with first term 11 and common difference -2
2. Evaluate $\sum_{n=1}^{50}(1+4n)$?
3. Find the sum of the integers between 1 and 2000 inclusive.
4. An arithmetic series begins $2+4+6+8+\dots$
   How many terms cause the sum to equal to 2550?

Answers:

1. 20th term: -27
2. 5150
3. 2 001 000
4. Solve $\frac{n}{2}(2a+(n-1)d)=2550$.
   Leads to $n^2+n-2550=0$.
   The positive solution: $n=50$ terms