Simpson's Rule

Numerical Methods 3

INU0114/514 (Maths 1)

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Simpson’s Rule

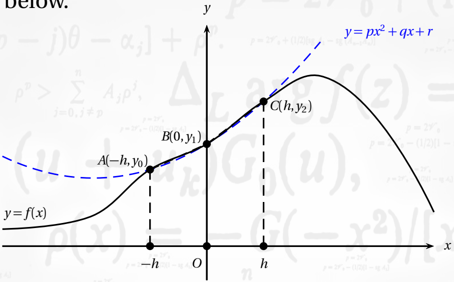
Recall that the *trapezium rule* approximates the shape of a function with a straight line (the top edge of the trapezium).

Simpson’s Rule is a better scheme, one in which the function shape is approximated by a section of a quadratic curve.

There are several variations of Simpson’s Rule each of which depends on the curve used to approximate the function curve. We will use the popular “composite Simpson’s Rule” which usually performs well even if the function being integrated is not smooth or lacks derivatives.

Derivation of Simpson's Rule

Simpson's Rule is based on the idea that the area under sections of the curve $y = f(x)$ can be approximated by the area under a quadratic passing through the section as shown below.



Suppose that a curve of the form $y = px^2 + qx + r$ passes through the points $A(-h, y_0)$, $B(0, y_1)$ and $C(h, y_2)$ as shown in the picture. The area under the quadratic curve between the ordinates is given by the integral:

$$\int_{-h}^{h} (px^2 + qx + r)\,\mathrm{d}x = \left[\frac{px^3}{3} + \frac{qx^2}{2} + rx\right]_{-h}^{h}$$

Evaluating the area integral:

$$\begin{aligned}
\int_{-h}^{h}(px^2+qx+r)\,\mathrm{d}x &= \left[\frac{px^3}{3}+\frac{qx^2}{2}+rx\right]_{-h}^{h} \\
&= \left(\frac{ph^3}{3}+\frac{qh^2}{2}+rh\right)-\left(-\frac{ph^3}{3}+\frac{qh^2}{2}-rh\right) \\
&= \frac{2ph^3}{3}+2rh \\
\int_{-h}^{h}(px^2+qx+r)\,\mathrm{d}x &= \tfrac{2}{3}h(ph^2+3r) \qquad (1)
\end{aligned}$$

Since points A, B and C are on the quadratic curve then we can write down the following three equations for those points:

$$\begin{aligned}
y_0 &= ph^2-qh+r \qquad (2)\\
y_1 &= r \qquad (3)\\
y_2 &= ph^2+qh+r \qquad (4)
\end{aligned}$$

Adding equations 2 and 4 gives

$$\begin{aligned}
y_0+y_2 &= 2ph^2+2r\\
y_0+y_2 &= 2ph^2+2y_1
\end{aligned}$$

Therefore

$$\begin{aligned}
y_0+y_2-2y_1 &= 2ph^2\\
\therefore \tfrac{1}{2}(y_0+y_2-2y_1) &= ph^2
\end{aligned}$$

$$\frac{1}{2}(y_0 + y_2 - 2y_1) = ph^2$$

We can substitute this for the ph^2 term in equation 1 to get:

$$\begin{aligned}
\int_{-h}^{h}(px^2+qx+r)\,\mathrm{d}x &= \tfrac{2}{3}h\left[\tfrac{1}{2}(y_0+y_2-2y_1)+3c\right] \\
&= \tfrac{2}{3}h\left[\tfrac{1}{2}(y_0+y_2-2y_1)+3y_1\right] \\
&= \tfrac{2}{3}h\left[\tfrac{1}{2}y_0+\tfrac{1}{2}y_2-y_1+3y_1\right] \\
&= \tfrac{2}{3}h\left[\tfrac{1}{2}y_0+\tfrac{1}{2}y_2+2y_1\right] \\
\int_{-h}^{h}(px^2+qx+r)\,\mathrm{d}x &= \tfrac{1}{3}h\left[y_0+4y_1+y_2\right]
\end{aligned}$$

We've managed to express the area under the curve in terms of the values of the ordinates y_0, y_1 and y_2 and with the strip width h. Therefore

$$\int_{a}^{b}(px^2+qx+r)\,\mathrm{d}x = \tfrac{1}{3}h\left[y_0+4y_1+y_2\right]$$

In this case we have $f(x) = px^2 + qx + r$ so that means $f(a) = y_0$, $f(a+h) = y_1$ and $f(a+2h) = f(b) = y_2$.

To obtain an approximation for the definite integral $\int_a^b f(x)\,\mathrm{d}x$ we divide the interval from a to b into an *even number of strips* (of which there are n) each of width h.

We assume that the top of each strip approximates a quadratic curve.

If $y_0, y_1, y_2, \ldots, y_n$ are successive ordinates at the end of each strip then

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &\approx \tfrac{1}{3}h\big[(y_0+4y_1+y_2)+(y_2+4y_3+y_4)+\ldots+(y_{n-2}+4y_{n-1}+y_n)\big]\\ &\approx \tfrac{1}{3}h\big[y_0+4y_1+2y_2+4y_3+2y_4+\ldots+2y_{n-2}+4y_{n-1}+y_n)\big]\end{aligned}$$

Factoring the odd and even terms in the brackets gives the formula we need:

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x \approx\ & \tfrac{1}{3}h\big[y_0+4(y_1+y_3+\ldots+y_{n-1})\\ & +2(y_2+y_4+\ldots+y_{n-2})+y_n\big]\end{aligned}$$

Simpson's Rule Formula

$$\int_a^b f(x)\,\mathrm{d}x \approx \tfrac{1}{3}h\big[y_0 + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2}) + y_n\big] \quad (5)$$

Where h is the strip width, given by $h = \dfrac{b-a}{n}$, and n is the number of strips. For the reasons outlined in the derivation, n must be even.

The brackets on the RHS contain the sums of the first, last and of odd ordinates and even strips. To aid memory, you can write the above equation as:

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{3}[\mathrm{F} + \mathrm{L} + (4 \times \mathrm{Odd}) + (2 \times \mathrm{Even})]$$

Where 'Odd' is the sum of the odd numbered ordinates and 'Even' is the sum of the even numbered ordinates.

Simpson's Rule

Use Simpson's Rule with 4 strips to evaluate $\int_0^4 3x^2\,\mathrm{d}x$

There are four strips so $n = 4$. $h = \frac{4-0}{4} = 1$.

We can construct a table of values for the function $y = 3x^2$.

x_n	0	1	2	3	4
y_n	0	3	12	27	48

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{1}{3}h\left[y_0 + 4(y_1 + y_3) + 2y_2 + y_4\right]$$

$$\int_0^4 3x^2\,\mathrm{d}x \approx \frac{1}{3}(1)[0 + 4(3 + 27) + 2(12) + 48]$$

$$\approx \frac{1}{3}(192)$$

$$\int_0^4 3x^2\,\mathrm{d}x \approx 64$$

The previous example gave the result $\int_0^4 3x^2\,\mathrm{d}x \approx 64$.

We can actually integrate this expression to find its value:

$$\int_0^4 3x^2\,\mathrm{d}x = \left[x^3\right]_3^4 = 4^3 - 0^3 = 64$$

In this case Simpson's rule gave the exact value.

In fact, Simpson's rule will give the exact value for polynomial functions up and including degree 3.

In all other cases there will be an error associated with the result.

Simpson's Rule

Use Simpson's Rule with 5 ordinates to evaluate $\int_0^1 \frac{1}{1+x}\,\mathrm{d}x$

Five ordinates means four strips so $n=4$. Therefore $h=\frac{1-0}{4}=\frac{1}{4}$.

We can construct a table of values for the function $y=\dfrac{1}{1+x}$.

x_n	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y_n	1	$\frac{4}{5}$	$\frac{2}{3}$	$\frac{4}{7}$	$\frac{1}{2}$

$$\begin{aligned}\int_0^1 \frac{1}{1+x}\,\mathrm{d}x &\approx \tfrac{1}{3}h\left[y_0+4(y_1+y_3)+2y_2+y_4\right]\\ &\approx \tfrac{1}{3}(\tfrac{1}{4})\left[1+4(\tfrac{4}{5}+\tfrac{4}{7})+2(\tfrac{2}{3})+\tfrac{1}{2}\right]\\ &\approx \tfrac{1}{12}\left[\tfrac{1747}{210}\right]\\ &\approx \tfrac{1747}{2520}\end{aligned}$$

To an accuracy of 6 decimal places we have $\int_0^1 \frac{1}{1+x}\,\mathrm{d}x \approx 0.693254$.

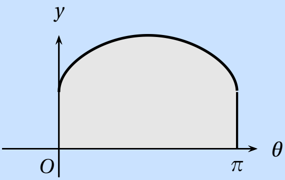
How accurate is this answer? Well, in this case we can find the exact value because we can integrate this function exactly:

$$\begin{aligned}\int_0^1 \frac{1}{1+x}\,\mathrm{d}x &= [\ln|1+x|]_0^1 \\ &= \ln 2 - \ln 1 \\ &= \ln 2 \\ &= 0.693147 \text{ (6 DP)}\end{aligned}$$

So the approximation obtained by Simpson's Rule is correct to 3 decimal places (the actual error is 0.015%). As with the trapezium rule, we could obtain a better estimate by increasing the number of strips.

Simpson's Rule

The following picture shows the curve $y = 1 + \sqrt{\sin\theta}$.



Use Simpson's Rule with six strips to find the area of shaded region bounded by the coordinate axes and the line $\theta = \pi$. Give the answer to four decimal places.

Six strips means $n = 6$ and so $h = \frac{\pi - 0}{6} = \frac{\pi}{6}$.

We can construct a table of values for the function $y = 1 + \sqrt{\sin\theta}$.

(Remember to use radians).

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
y	1	1.70711	1.93060	2	1.93060	1.70711	1

(Intermediate calculations are given to 5 decimal places here).

Using the informal version of Simpson's Rule:

$$\int_a^b f(x)\,\mathrm{d}x \quad \approx \quad \frac{h}{3}[\mathrm{F}+\mathrm{L}+(4\times\mathrm{O})+(2\times\mathrm{E})]$$

Substituting the values from the table:

$$\begin{aligned}
\therefore \int_0^\pi \left(1+\sqrt{\sin\theta}\right)\mathrm{d}\theta \quad &\approx \quad \tfrac{1}{3}\tfrac{\pi}{6}[1+1+4(1.70711+2+1.70711) \\
&\qquad +2(1.93060+1.93060)] \\
&\approx \quad \tfrac{\pi}{18}[2+4(5.41422)+2(3.8612)] \\
&\approx \quad \tfrac{\pi}{18}(31.37928) \\
\int_0^\pi \left(1+\sqrt{\sin\theta}\right)\mathrm{d}\theta \quad &\approx \quad 5.4767
\end{aligned}$$

Therefore, the shaded region has an area of 5.4767 units^2.

Trapezium rule and Simpson's rule compared

Consider the integral

$$\int_0^6 \frac{1}{1+x}\,\mathrm{d}x$$

Use the Trapezium rule and Simpson's rule with six strips and calculate the percentage error in each case.

Six strips over the interval means $h = \frac{6-0}{6} = 1$.

Construct a table of values for use with both methods:

x_n	0	1	2	3	4	5	6
y_n	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$

Trapezium rule first:

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &\approx \frac{h}{2}\left[\mathrm{F}+\mathrm{L}+2(\text{everything else})\right]\\ &\approx \tfrac{1}{2}\left[1+2(\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\tfrac{1}{6})+\tfrac{1}{7}\right]\\ &\approx \tfrac{1}{2}\times\tfrac{283}{70}\\ \therefore \int_0^6 \frac{1}{1+x}\,\mathrm{d}x &\approx \tfrac{283}{140}\end{aligned}$$

Using the same table of values:

x_n	0	1	2	3	4	5	6
y_n	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$

We can apply Simpson's rule:

$$\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x &\approx \frac{h}{3}[\mathrm{F}+\mathrm{L}+(4\times \mathrm{O})+(2\times \mathrm{E})] \\
\int_0^6 \frac{1}{1+x}\,\mathrm{d}x &\approx \tfrac{1}{3}\left[1+\tfrac{1}{7}+4(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6})+2(\tfrac{1}{3}+\tfrac{1}{5})\right] \\
&\approx \tfrac{1}{3}(\tfrac{8}{7}+\tfrac{11}{3}+\tfrac{16}{15}) \\
&\approx \tfrac{617}{315}
\end{aligned}$$

Is Trapezium rule or Simpson's rule more accurate? Find the percentage errors compared to the true value....

Direct integration gives the true value:

$$\int_0^6 \frac{1}{1+x}\,\mathrm{d}x = [\ln(1+x)]_0^6 = \ln 7$$

The percentage error is calculated from

$$\% \text{ error} = \frac{\text{Absolute error}}{\text{True value}} \times 100$$

For the Trapezium rule we have:

$$\% \text{ error} = \frac{\left|\frac{283}{140} - \ln 7\right|}{\ln 7} \times 100 \approx 3.8\%$$

For Simpson's rule we have

$$\% \text{ error} = \frac{\left|\frac{617}{315} - \ln 7\right|}{\ln 7} \times 100 \approx 0.7\%$$

In this case (and generally) Simpson's rule gives a more accurate answer than Trapezium rule.

Test yourself...

You should be able to solve the following problems using Simpson's rule if you have understood these notes.

1. Use $h = 1$ to evaluate $\int_{-2}^{2} (4 - x^2)\,\mathrm{d}x$
2. Find the exact value of the integral and hence find the percentage error in the estimate in (1).
3. Use six strips to evaluate $\int_{1}^{2} (\sin x)^x\,\mathrm{d}x$ to 4 decimal places.
4. Explain how you could improve the accuracy of your answer to (3).

Answers:

1. Simpson's rule gives $\frac{32}{3}$.
2. Exact value is also $\frac{32}{3}$. There is no error in the estimate.
3. 0.9412 (to 4 decimal places). You must use radians to get this.
4. By using more strips (a smaller value of h). It isn't possible to get an answer to this using the rules of integration; we must use a numerical method.