Fixed Point Iteration

Numerical Methods 2

INU0114/514 (Maths 1)

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Objectives

We continue with our investigation of numerical solution of equations with one final method:

- Fixed Point Iteration method
- Convergence of FPI and cobweb diagrams.

FPI uses a rearrangement of the given equation along with an initial estimate of the solution (obtained via a graph or by bracketing).

Solving equations by iteration

Consider the recurrence relation

$$x_{n+1} = \cos x_n \, , \; x_0 = 1$$

Calculate the values of $x_1, x_2, \ldots$ and observe what happens. Switch to radian mode before you begin!

The numbers in the sequence converge. After about 20 iterations we find

$$x = 0.739 \quad \text{(to 3 D.P.)}$$

We saw in our study of recurrence relations that convergence means $x_n \longrightarrow x_{n+1}$ as $n \longrightarrow \infty$.

This calculated value is a solution to the equation

$$x = \cos x$$

and the recurrence relation provided the solution.

Will this work for other equations for which we can't rearrange to find x explicitly?

Fixed Point Iteration

To find a solution to the equation

$$f(x) = 0$$

We rearrange the equation into the form $x = g(x)$ and then set up an iterative scheme (a recurrence relation) based on it

$$\boxed{x_{n+1} = g(x_n)}$$

and using an initial estimate x_0 to begin the iterations.

As we shall see there may be several possible ways to rearrange the equation and some rearrangements may not lead to successful solution of the equation.

Solution by fixed point iteration

The equation

$$x^3 - \sin x - 1 = 0$$

has a solution $x \approx 1$. Find the solution to 4 decimal places.

There are many possible rearrangements. Let's try

$$x^3 = 1 + \sin x \qquad \Rightarrow x = (1 + \sin x)^{\frac{1}{3}}$$

The iterative scheme will be

$$x_{n+1} = (1 + \sin x_n)^{\frac{1}{3}} \qquad , x_0 = 1$$

Important: the value of x is an angle in radians.

Given

$$x_{n+1} = (1 + \sin x_n)^{\frac{1}{3}} \qquad , x_0 = 1$$

The iterations, recorded to four decimal places, are:

$$\begin{aligned} x_1 &= 1.2257 \\ x_2 &= 1.2474 \\ x_3 &= 1.2489 \\ x_4 &= 1.2490 \\ x_5 &= 1.2491 \\ x_6 &= 1.2491 \end{aligned}$$

After 6 iterations the solution has been calculated to be

$$x = 1.2491$$

Would a different rearrangement also have worked?

An investigation

Consider the equation

$$x^2 - 4x + 2 = 0$$

Examine the convergence for different rearrangements and with an initial estimates of $x_0 = 3$ and $x_0 = 4$.

Two possible rearrangements of this equation are

$$x = \sqrt{4x - 2} \quad \text{and } x = \frac{1}{4}(x^2 + 2)$$

These equations obviously lead to the iterative schemes:

$$x_{n+1} = \sqrt{4x_n - 2} \quad \text{and } x_{n+1} = \frac{1}{4}(x_n^2 + 2)$$

The following slide shows the sequence of iterates for each scheme.

$x_{n+1} = \sqrt{4x_n - 2}$	
$x_0 = 3$	$x_0 = 4$
3	4
3.162	3.742
3.263	3.601
3.325	3.522
3.361	3.477
3.383	3.451
3.396	3.435
3.403	3.427
3.408	3.421
3.411	3.418
3.412	3.417
3.413	3.416
3.413	3.415

With this rearrangement, both starting values seem to be converging on the same solution.

$x_{n+1} = \frac{1}{4}(x_n^2 + 2)$	
$x_0 = 3$	$x_0 = 4$
3	4
2.75	4.5
2.391	5.563
1.929	8.235
1.430	17.45
1.011	76.67
0.756	1470.1
0.643	540320
0.603	7×10^{10}
0.591	10^{21}
0.587	10^{41}
0.586	10^{82}
0.586	10^{164}

Here, one starting value leads to convergence (but a different solution to the first) but the second value diverges.

Analysing convergence with a graph

Consider again the equation $x^2 - 4x + 2 = 0$.

The two rearrangements we considered were

$$x = \sqrt{4x - 2} \quad \text{and } x = \tfrac{1}{4}(x^2 + 2)$$

Leading to the iterative schemes:

$$x_{n+1} = \sqrt{4x_n - 2} \quad \text{and } x_{n+1} = \tfrac{1}{4}(x_n^2 + 2)$$

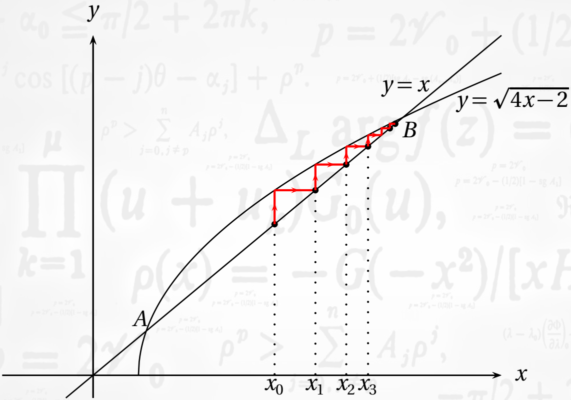
Let's investigate which rearrangement and starting values will be successful.

We will plot $y = x$ and $y = g(x)$ on the same graph and observe what happens as we iterate from some starting value.

Fixed point iteration takes an estimate x and uses the function $g(x)$ to generate a new value y.

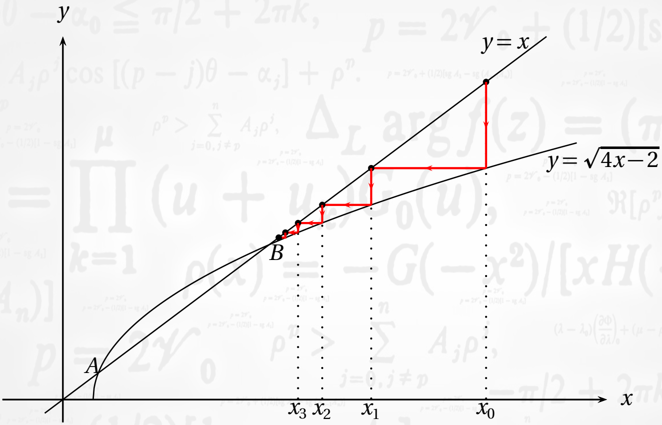
On a graph this is equivalent to starting at a point on the straight-line $y = x$ and moving vertically to the curve. That function value y becomes the next x value (equivalent to moving horizontally to the straight line).

Let's examine the first rearrangement. From an initial estimate x_0 we can visualise the iterations:



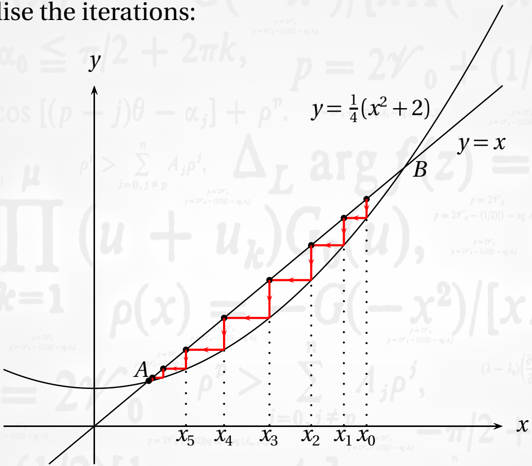
Provided x_0 is somewhere between A and B then the iterates will converge to the solution at B. This is an example of a *staircase* convergence.

Let's examine the first rearrangement. From an initial estimate x_0 we can visualise the iterations:



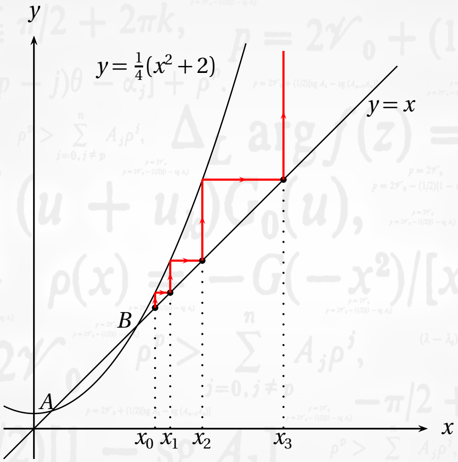
The graph shows that for any x_0 bigger than the root at B that convergence will to the root at B will happen. This is staircase convergence again.

Let's examine the second rearrangement. From an initial estimate x_0 we can visualise the iterations:



For this value of x_0 the graph shows staircase convergence to the solution at A.

Let's change the initial estimate x_0 — move it to right of solution B. The iterations behave like this:

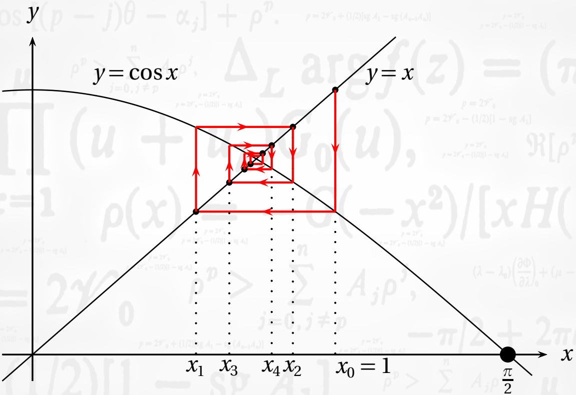


For this value of x_0 the graph shows the iterates will increase quickly. This is an example of a *staircase divergence*.

Cobweb diagram

Examine the convergence of $\cos x - x = 0$ using a graph with $x_0 = 1$.

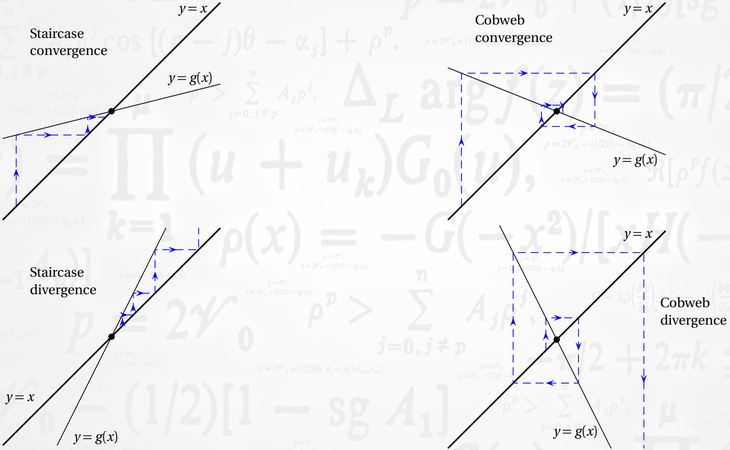
The equation can be rearranged to give $x = \cos x$ and the graph looks like this:



The graph shows how the iterates x_0, x_1, x_2... oscillate around the actual solution, but that convergence is also taking place. This type of behaviour is called *cobweb convergence*.

Cobwebs and staircases

The intersection of $y = x$ and $y = g(x)$ on a graph, when magnified, is always one of the four possible pictures below.



Notice how the only quantity changing is the slope of $g(x)$. It is $g'(x)$ near the root which determines whether convergence happens.

Convergence conditions (optional!)

An iteration of the form $x_{n+1} = g(x_n)$ converges when the gradient of $y = g(x)$ at the point of intersection with the line $y = x$ satisfies the condition $\left|g'(x)\right| < 1$, provided a suitable value for x_0 is chosen.

The 'suitable value for x_0' means a value is chosen in the neighbourhood of the intersection.

The equation $x^2 - 4x + 2 = 0$ has two solutions at $x = 3.4142$ and $x = 0.5858$.

One rearrangement was $x_{n+1} = \frac{1}{4}(x_n^2 + 2)$. In this case $g(x) = \frac{1}{4}(x^2 + 2)$ and therefore $g'(x) = \frac{1}{2}x$. Testing the solutions:

$$g'(3.4142) = 1.707 > 1 \quad \therefore \text{Convergence to this root IS NOT possible}$$

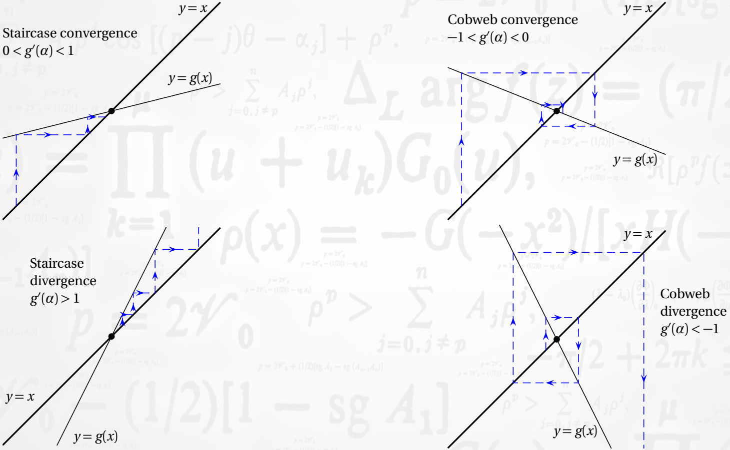
And

$$g'(0.5858) = 0.293 < 1 \quad \therefore \text{Convergence to this root IS possible}$$

The problem is we don't know these values before we start! So a good initial estimate of the root is required before applying this test.

Cobwebs and staircases

Let's revisit those cobweb and staircases and state the conditions on $g'(\alpha)$ which leads to each of them.



In each case the root is located at $x = \alpha$.

Determining convergence

The equation $x^3 - x - 2 = 0$ has a solution $x \approx 1.5$. Find out whether the arrangement

$$x_{n+1} = x_n^3 - 2 \ , x_0 = 1.5$$

will lead to convergence.

In this case we have $g(x) = x^3 - 2$ so that

$$g'(x) = 3x^2 \qquad \therefore g'(1.5) = 6.75$$

Since $g' > 1$ then this scheme will likely lead to staircase divergence. See the diagrams on the previous slide.

We can never be 100% sure of our conclusions because we calculated $g'(x_0)$ whereas the behaviour is actually determined by $g'(\alpha)$, where $x = \alpha$ is the value of the root (which is unknown!)

The closer the estimate x_0 is to α the more we can trust the conclusion from $g'(x_0)$!

Summary

- Fixed point iteration solves the equation $f(x) = 0$ rearranging the equation into a new form $x = g(x)$.
- An iterative scheme is made using the recurrence relation $x_{n+1} = g(x_n)$.
- The method can be unreliable if no analysis of the $x = g(x)$ is carried out. A graph of $y = x$ against $y = g(x)$ can be very helpful in determining which rearrangements and initial values will produce solutions. Staircase convergence or divergence, or cobweb (oscillatory) convergence or divergence can be demonstrated.
- Convergence is dependent on the derivative $g'(x)$ and is assured if $\left|g'(x)\right| < 1$ for a well chosen initial value x_0.