NEWTON-RAPHSON METHOD

NUMERICAL METHODS 1

INU0114/514 (MATHS 1)

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Objectives

In this session we'll introduce Newton-Raphson method (also called just Newton's Method) getting accurate solutions to nonlinear equations.

- Graphical estimation of a solution
- Derivation of the Newton-Raphson method
- Solving equations with Newton-Raphson method
- Limitations and failure of the method

Newton-Raphson method uses an initial estimate of the solution and the first derivative of the function to obtain a more accurate estimate.

Since the next estimate depends on the previous estimate then Newton-Raphson method is in the form of a recurrence relation.

Graphical estimation of a solution

We saw in an earlier presentation how the Intermediate Value Theorem is helpful for bracketing a root.

Another useful technique to is use a graph to find an initial estimate of the solution.

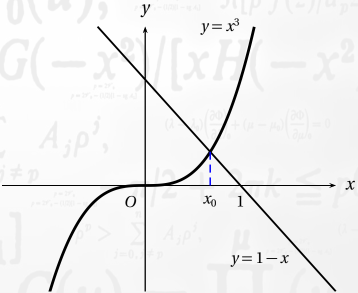
We might do this by rearranging the equation and plotting the two sides of the new equation on the same axes.

For example let's estimate the solution of

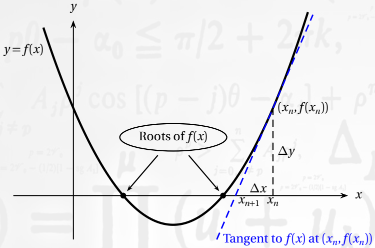
$$x^3 + x - 1 = 0$$

Rearrange the equation to make $x^3 = 1 - x$.

The solution to the equation is represented on a graph by the intersection of $y = x^3$ and $y = 1 - x$.



Newton-Raphson derivation



The picture shows how the current estimate of the root x_n can be used to find a closer estimate x_{n+1}.

The process starts with an initial estimate x_0 which might be found using bracketing or a graphical method.

Given an estimate x_n we can evaluate the function value as $f(x_n)$. A tangent line to this point intersects the x-axis at point x_{n+1} which is should be closer to the root.

The gradient of the tangent is given by:

$$f'(x_n) = \frac{\Delta y}{\Delta x} = \frac{f(x_n)}{x_n - x_{n+1}}$$

This can be rearranged to give:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad (1)$$

This is the Newton-Raphson method; starting with an initial estimate x_0 the numerical scheme above can be iterated until we have a solution to a specified accuracy.

Solution with Newton-Raphson

Given that the equation $x^2 - 2 = 0$ has a solution in $(1, 2)$, use Newton-Raphson to obtain a solution to six decimal places.

Let $f(x) = x^2 - 2$. Then the derivative is $f'(x) = 2x$.

Newton-Raphson method uses $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

The iterative scheme for solving the equation is $x_{n+1} = x_n - \dfrac{x_n^2 - 2}{2x_n}$.

Let's use $x_0 = 1.5$ to calculate x_1

$$x_1 = 1.5 - \frac{1.5^2 - 2}{2(1.5)} = 1.41666666\ldots$$

Repeat to find x_2, x_3 and so on.

$$\begin{aligned} x_2 &= 1.414217 \\ x_3 &= 1.414214 \\ x_4 &= 1.414214 \end{aligned}$$

Note that the values were calculated using the full accuracy of the calculator even though only six decimal places were written down.

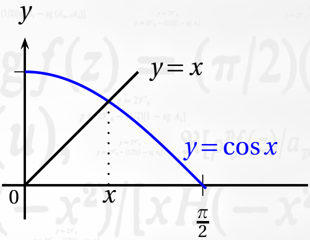
Solution with Newton-Raphson

Solve the equation $\cos x = x$ using Newton-Raphson method to six decimal places.

We need an initial estimate of the solution so consider the graph of the functions $y = x$ and $y = \cos x$.

The solution x is the intersection of the curves and there is clearly a solution in $(0, \frac{\pi}{2})$

Since $\frac{\pi}{2} \approx 1.57$, let's take $x_0 = 1$.



Let $f(x) = \cos x - x = 0$ so that $f'(x) = -\sin x - 1$

Newton-Raphson method for this equation is

$$x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1} \Rightarrow x_{n+1} = x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}$$

$$x_{n+1} = x_{n+1} = x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}$$

Starting with $x_0 = 1$ the iterates are:

$$\begin{aligned} x_1 &= 0.750364 \\ x_2 &= 0.739113 \\ x_3 &= 0.739085 \\ x_4 &= 0.739085 \end{aligned}$$

Convergence to six decimal places is achieved. The solution is $x = 0.739085$.

The behaviour of Newton-Raphson

Consider the equation $x^3 - 2x + 2 = 0$. Investigate the behaviour of Newton-Raphson method with $x_0 = 0$.

Let $f(x) = x^3 - 2x + 2$. The gradient is $f(x) = 3x^2 - 2$.

Newton-Raphson iteration scheme is:

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n + 2}{3x_n^2 - 2}$$

Starting with $x_0 = 0$ we get $x_1 = 1$.

The next iteration gives $x_2 = 0$.

The iterates continue to cycle between 0 and 1. No solution is reached.

Limitations of the method

Newton-Raphson method can be a very efficient method for finding the zero of a function. However, there are situations where the method will fail.

Consider what would have happened if the initial estimate x_0 corresponds to a stationary point of the function. In this case, the tangent line is parallel to the axis, so $f'(x) = 0$. We are unable to compute x_1.

For some functions, some starting points may enter an infinite cycle, preventing convergence.

In general, the behavior of the sequence can be very complex.