Shifted SHM

Kinematics 2

INU0115/515 (Maths 2)

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Objectives

This is the final presentation of the kinematics topic. We're going look at some more examples of SHM, but this time the oscillation does not take place about an origin.

For example: we have a particle but the distance from an origin varies between 5 cm and 10 cm with SHM.

It's only a small change but it allows us to model a wider range of situations. For example:

- The rise and fall of water in a harbour.
- The approximate variation in the amount of daylight at a location.
- Pulsations of variable stars.
- ...and more!

Let's get started!

SHM recap

We saw that a particle oscillating about an origin with SHM has an equation of motion described by

$$\boxed{\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x}$$

where x is the position at time t and ω^2 is a positive constant.

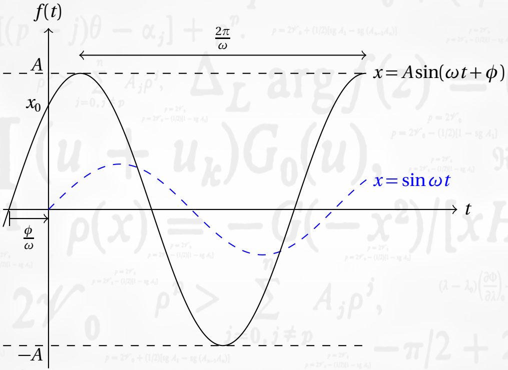
The solution to the differential equation is obtained by integration and may be expressed in a number of equivalent ways. For example:

$$\boxed{x = A(\sin \omega t + \phi)}$$

where A and ϕ are constants which can be determined by boundary (or initial) conditions.

Displacement-time graph (general case)

The general solution to the SHM equation is a sine wave.



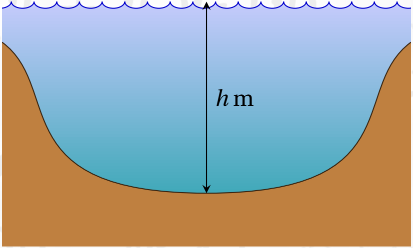
The key property we notice here is that the sine wave oscillates about the horizontal axis (the line $x = 0$).

SHM with shifted origin

SHM about an origin is the easiest case to analyse but it isn't always the most natural for a given situation.

Consider a simple model of water height in a harbour. The height changes as the tide comes in and out. Suppose we have formula which states the water height h metres at time t hours is

$$h = 4 + 2\cos(\tfrac{\pi}{6}t) \tag{1}$$



First: is this a simple harmonic motion problem?

We can check by differentiating (using chain rule) with respect to t

$$\frac{dh}{dt} = \frac{\pi}{6} \times \left(-2\sin\left(\tfrac{\pi}{6}t\right)\right)$$

And again...

$$\frac{d^2h}{dt^2} = -\left(\frac{\pi}{6}\right)^2 2\cos\left(\tfrac{\pi}{6}t\right)$$

Comparing to equation (1) we can see this is the same as

$$\frac{d^2h}{dt^2} = -\omega^2(h-4)$$

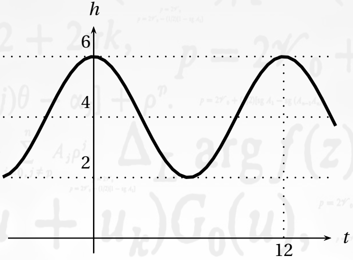
where $\omega = \frac{\pi}{6}$.

This *is* an equation for SHM; the only difference is that the origin of motion has been shifted to the point $h = 4$. We could change the variable:

$$\frac{d^2H}{dt^2} = -\omega^2 H$$

where $H = h - 4$, to make it more obvious!

You can see this clearly on the graph of $h = 4 + 2\cos(\frac{\pi}{6}t)$.



The period is $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\pi/6} = 12$ hours.

You can see from the graph:

- the water level rises and falls about a mean value of 4 metres every 12 hours. In other words — the period of the system is 12 hours.
- the amplitude of oscillation is 2 metres.

In a system where SHM occurs about a point not at the origin then the displacement equation will have the general form:

$$\boxed{y = A + B\sin\omega t \quad \text{or} \quad y = A + B\cos\omega t}$$

Where:

- the oscillation is about the point $y = A$ (and it begins there at $t = 0$).
- the amplitude of the oscillations is B.

and the angular frequency ω has its usual meaning.

Shifted SHM

At midday the Sun is highest in the sky and measured to be x degrees above the horizon. From Kuwait City the midday elevation of the Sun is

$$x = 60 + 23 \sin \omega t$$

where t is the number of days since the vernal (Spring) equinox. The period of oscillation is 365 days. (a) Find the maximum and minimum values of x and (b) the values of t for which they occur during the year.

(a) First — the maximum and minimum values.

The oscillations vary like $x = 60 \pm 23$ (since $-1 \leq \sin \omega t \leq 1$).

Therefore

$$x_{max} = 83 \quad \text{and} \quad x_{min} = 37$$

(b) Next, we need to find ω. The definition is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{365}$$

The SHM equation is therefore

$$x = 60 + 23\sin\left(\frac{2\pi}{365}t\right)$$

The maximum and minimum values occur when the sine function reaches extreme values. We must solve

$$\sin\left(\frac{2\pi}{365}t\right) = 1 \quad \text{and} \quad \sin\left(\frac{2\pi}{365}t\right) = -1$$

The first has a solutions at $\frac{\pi}{2}$ and the second at $\frac{3\pi}{2}$.

$$\frac{2\pi}{365}t = \frac{\pi}{2} \quad \text{and} \quad \frac{2\pi}{365}t = \frac{3\pi}{2}$$

Solve for t:

$$t = \frac{365}{4} \quad \text{and } t = \frac{1095}{4}$$

Maximum elevation occurs at $t \approx 91$ days and minimum elevation at $t \approx 274$ days (after the vernal equinox).