Circular Motion

Kinematics 1

INU0115/515 (Maths 2)

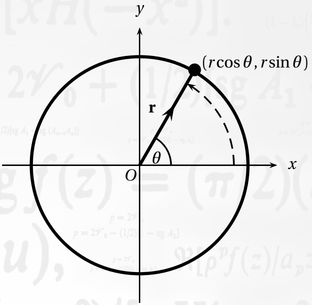
Dr Adrian Jannetta MIMA CMath FRAS

Objectives

- In this presentation we'll examine a simple situation in which a particle is being made to move along a circle with uniform angular speed.
- We'll use our knowledge of vectors and calculus to derive equations describing the speed and acceleration of the particle.
- We'll demonstrate that motion in a circle must be caused by a centrally directed force.
- In the final part of the course we'll see that circular motion is related to a more general type of oscillation called *simple harmonic motion*.

Circular motion

Consider a particle being made to move on a circle of radius r with uniform angular speed ω.



At time t the particle is located at the point $(r\cos\theta, r\sin\theta)$.

The angle θ radians is dependent on the angular speed:

$$\theta = \omega t$$

Since there are 2π radians in a full circle, the period T taken for the particle to make 1 revolution is equal to

$$\boxed{T = \frac{2\pi}{\omega}} \tag{1}$$

Position, velocity, acceleration vectors

Since the particle is located at $(r\cos\omega t, r\sin\omega t)$ then the position vector of the particle is

$$\mathbf{r} = r\cos\omega t\mathbf{i} + r\sin\omega t\mathbf{j}$$

Differentiate this with respect to t to obtain the velocity vector: and acceleration vectors:

$$\mathbf{v} = -\omega r\sin\omega t\mathbf{i} + \omega r\cos\omega t\mathbf{j}$$

Differentiate again to get the acceleration vector:

$$\mathbf{a} = -\omega^2 r\cos\omega t\mathbf{i} - \omega^2 r\sin\omega t\mathbf{j}$$

We're going to play with these vectors now. Let's see what the mathematics tells us about circular motion!

Speed (and acceleration)

Let's find the speed of the particle: it is the magnitude of the velocity vector:

$$\begin{aligned} v &= |\mathbf{v}| \\ &= \sqrt{(-\omega r \sin \omega t)^2 + (\omega r \cos \omega t)^2} \\ &= \sqrt{\omega^2 r^2 (\sin^2 \omega t + \cos^2 \omega t)} \\ &= \sqrt{\omega^2 r^2 (1)} \end{aligned}$$

Therefore the (linear) speed of the particle is

$$\boxed{v = \omega r} \tag{2}$$

Now let's calculate the magnitude of the acceleration vector:

$$\begin{aligned} a &= |\mathbf{a}| \\ &= \sqrt{(-\omega^2 r \cos \omega t)^2 + (-\omega^2 r \sin \omega t)^2} \\ &= \sqrt{\omega^4 r^2 (\cos^2 \omega t + \sin^2 \omega t)} \end{aligned}$$

Since $\cos^2 \omega t + \sin^2 \omega t \equiv 1$ the acceleration is constant and equal to

$$\boxed{a = \omega^2 r} \tag{3}$$

Scalar product

The angle between the velocity and acceleration vectors can be calculated using the scalar product formula:

$$\cos\theta = \frac{\mathbf{v}.\mathbf{a}}{va}$$

Calculation of the dot product gives:

$$\begin{aligned}
\mathbf{v}.\mathbf{a} &= (-\omega r \sin\omega t\mathbf{i} + \omega r \cos\omega t)\cdot(-\omega^2 r \cos\omega t\mathbf{i} - \omega^2 r \sin\omega t\mathbf{j}) \\
&= (-\omega r \sin\omega t)(-\omega^2 r \cos\omega t) + (\omega r \cos\omega t)(-\omega^2 r \sin\omega t) \\
&= \omega^3 r \sin\omega t \cos\omega t - \omega^3 r \sin\omega t \cos\omega t \\
\therefore \mathbf{v}.\mathbf{a} &= 0
\end{aligned}$$

This result implies that $\cos\theta = 0$ and that the angle between the vectors is $\theta = 90°$ — they are always perpendicular.

Centripetal acceleration

Consider the acceleration vector again:

$$\begin{aligned} \mathbf{a} &= -\omega^2 r \cos \omega t \mathbf{i} - \omega^2 r \sin \omega t \mathbf{j} \\ &= -\omega^2 (r \cos \omega t \mathbf{i} + r \sin \omega t \mathbf{j}) \\ \therefore \mathbf{a} &= -\omega^2 \mathbf{r} \end{aligned}$$

This shows that the acceleration vector is a constant multiple of the position vector. It means the two vectors are parallel, with the negative sign indicating opposing directions.

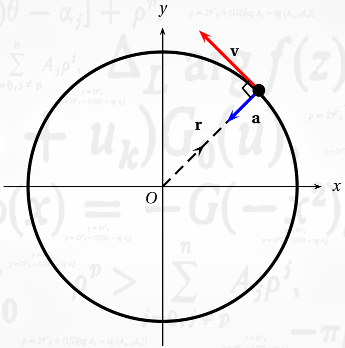
Therefore the acceleration vector is directed towards the origin — the centre of the circle.

We call it *centripetal acceleration*. By Newton's second law, there must be a force acting on the particle towards the centre.

Analysing this system with vector methods has generated simple formulae which we can use to answer questions about uniform motion in a circle.

Circular motion

This final diagram shows the relationship between the position, velocity and acceleration vectors for a particle moving around a circle with uniform angular speed.



Visit `https://goo.gl/oex8rl` to see an interactive version of this.

Equations of motion

The following equations apply to a particle moving around a circle of radius r metres with constant angular speed ω rad s^{-1}.

The period — time (in seconds) taken to make one revolution — is given by:

$$\boxed{T = \frac{2\pi}{\omega}}$$

The linear speed (in m s^{-1}) of the particle is:

$$\boxed{v = \omega r}$$

The magnitude of the acceleration (m s^{-2}) is

$$\boxed{a = \omega^2 r}$$

Let's tackle a couple of simple problems with these equations. It's also useful to know that the circumference of a circle is $2\pi r$.

Test yourself...

Let's try a few problems about circular motion.

1. Assuming the Earth rotates once in a 24 hour period calculate the angular speed ω rad s^{-1}.
2. Calculate the linear speed of a point on the Earth's equator. You may assume Earth to be a sphere with a radius of 6371 km. Give your answer in km s^{-1}.
3. A satellite is moving at 8 km s^{-1} in a circular orbit around a planet. If the radius of the satellite orbit satellite is 6800 km calculate the period (in minutes) of the orbit and the acceleration (in m s^{-2}).
4. The position vector of a particle is

$$\mathbf{r} = (3\sin 2t)\mathbf{i} + (3\cos 2t + 1)\mathbf{j}$$

Find the Cartesian equation of the circle.

Answers:

1. $\omega = 7.27 \times 10^{-5}$ rad s^{-1} (to 3 SF).
2. $v = 0.463$ km s^{-1}.
3. $T \approx 89.0$ minutes. $a \approx 9.142$ m s^{-2}.
4. $x^2 + (y-1)^2 = 9$.