Motion with variable acceleration

Kinematics 1

INU0115/515 (Maths 2)

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Objectives

- In the final part of this course we're going to look at an area of physics called *kinematics.*
- Kinematics is the study of particle motion under the action of forces.
- In your previous education you may studied this already — but with the simpler situation where acceleration is constant. (The resulting equations of motion are sometimes called *suvat* or *uvast*).
- We're going to apply our knowledge of vectors (from Maths 1) and calculus (from Maths 2) to analyse kinematical problems.

Motion with variable acceleration

When a particle moves with constant acceleration there are a number formulae which relate velocity, time, distance and acceleration (the UVAST equations).

For situations where acceleration is *not* constant we have to use calculus.

Velocity is the rate of change of distance s with time t. Acceleration a is the rate at which velocity varies. We can write these relationships as

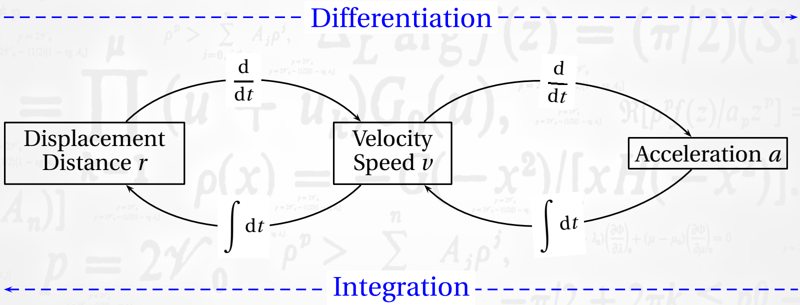
$$v = \frac{\mathrm{d}s}{\mathrm{d}t} \text{ and } a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

Since integration is the reverse of differentiation we find the velocity and distance from acceleration:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} \quad \Rightarrow v = \int a\,\mathrm{d}t$$

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} \quad \Rightarrow s = \int v\,\mathrm{d}t$$

Distance, speed, acceleration



Differentiating distance to get velocity and acceleration

A particle moves in a straight line so that its displacement s (in metres) from a fixed point O on the line is given by $s = t^3 - 3t^2 - 9t$.
(a) Find the velocity after t seconds. (b) Find the time when the velocity is zero.
(c) Find the acceleration after 3 seconds.

Part (a); differentiate s with respect to t to get velocity:

$$\begin{aligned} v &= \frac{ds}{dt} \\ &= \frac{d}{dt}(t^3 - 3t^2 - 9t) \\ \therefore v &= 3t^2 - 6t - 9 \end{aligned}$$

Part (b); solve the equation for $v = 0$, i.e.

$$\begin{aligned} 3t^2 - 6t - 9 &= 0 \\ t^2 - 2t - 3 &= 0 \\ (t-3)(t+1) &= 0 \end{aligned}$$

Therefore $t = 3$ seconds or $t = -1$ seconds. The negative value means the velocity was zero 1 second before the particle got to O.

Part (c); the acceleration is given by

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{d}{dt}(3t^2 - 6t - 9) \\ \therefore a &= 6t - 6 \end{aligned}$$

After 3 seconds, the acceleration is $a = 6 \times 3 - 6 = 12\,\text{m s}^{-2}$.

Integrating acceleration to get velocity and distance

A particle moving on a straight line has an initial velocity of $2\,\mathrm{m\,s^{-1}}$ at point O on the line. The particle moves so that t seconds later its acceleration is given by $(2t-6)\,\mathrm{m\,s^{-2}}$. Find expressions for the velocity and distance from O when $t=5$ seconds.

Integrate the acceleration to get velocity:

$$\begin{aligned} v &= \int a\,\mathrm{d}t \\ &= \int (2t-6)\,\mathrm{d}t \\ v &= t^2-6t+C_1 \end{aligned}$$

where C_1 is a constant. Given that $v=2$ when $t=0$:

$$2=0^2-6(0)+C_1 \quad \Rightarrow C_1=2$$

Therefore $v=t^2-6t+2$.

When $t=5$ we find

$$v=5^2-6(5)+2=-3\,\mathrm{m\,s^{-1}}$$

The negative sign indicates the particle is moving towards the left!

Integrate the velocity to get the displacement (from O)

$$\begin{aligned} s &= \int v\,\mathrm{d}t \\ &= \int (t^2-6t+2)\,\mathrm{d}t \\ s &= \tfrac{1}{3}t^3-3t^2+2t+C_2 \end{aligned}$$

where C_2 is a constant. Given $s=0$ when $t=0$ we see that $C_2=0$.

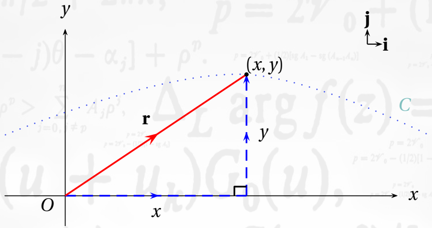
Therefore $s=\frac{1}{3}t^3-3t^2+2t$.

After 5 seconds $s=-23\frac{1}{3}$ m.

The negative sign indicates that the particle is to the left of the point O.

Position vector in 2D space

To track a particle moving through 2D space we need to study the changes in its position vector **r**.



$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

The length of the position vector $r = |\mathbf{r}| = \sqrt{x^2 + y^2}$ is the distance from O. (The letter 'r' is preferred to s in these situations).

We will examine cases where the **r** is a function of time t, i.e., where

$$x = f(t), \quad y = g(t)$$

These (parametric) equations define a set of points (x, y) over a given time interval that trace out a path C through space.

Recap: Vector magnitude and direction

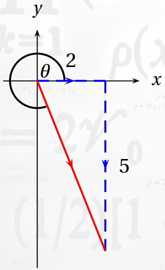
Calculating vector direction (2D)

A particle is moving with a velocity $\mathbf{v} = (2\mathbf{i} - 5\mathbf{j})\,\mathrm{m\,s^{-1}}$. Calculate the speed and direction (measured in degrees from the positive x-axis).

The speed of the particle is found from Pythagoras' Theorem:

$$v = |\mathbf{v}| = \sqrt{2^2 + (-5)^2} = \sqrt{29}\,\mathrm{m\,s^{-1}}$$

The direction is found using trigonometry; make a sketch of the vector beginning at the origin:



The angle θ, in the fourth quadrant, is:

$$\theta = \tan^{-1}\left(\frac{-5}{2}\right) = -68.2°$$

Add 360° to get the angle in the correct quadrant.

$$\theta = 291.8°$$

The speed of the particle is $\approx 5.4\,\mathrm{m\,s^{-1}}$ at an angle of 291.8° measured counter clockwise from the positive x-axis.

Differentiating the position vector

The position, velocity and acceleration vectors of a particle are related to one another by their rate of change with time.

Consider a particle with a position vector given by:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

The velocity vector is obtained by differentiating the position vector with respect to t.

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\mathbf{j}$$

At any time t the magnitude of $\mathbf{v}$ is the particle's speed. The direction of $\mathbf{v}$ is the direction of motion.

The acceleration vector is obtained by differentiating each component of the velocity with respect to t.

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathbf{i} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\mathbf{j}$$

Newton's dot notation is often used to denote these quantities. So the velocity and acceleration are sometimes written as

$$\mathbf{v} = \dot{\mathbf{r}} \quad \text{and} \quad \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$$

Finding the position, velocity and acceleration vectors

A particle has a position vector given by

$$\mathbf{r} = t^3\mathbf{i} + 6t^2\mathbf{j} - 3\mathbf{k}$$

Find the position, velocity and acceleration vectors when $t = 2$ seconds.

The position vector at $t = 2$ is found using a quick substitution:

$$\begin{aligned} \mathbf{r} &= 2^3\mathbf{i} + 6(2^2)\mathbf{j} - 3\mathbf{k} \\ &= 8\mathbf{i} + 24\mathbf{j} - 3\mathbf{k} \end{aligned}$$

Differentiate $\mathbf{r}$ with respect to time and substitute $t = 2$

$$\begin{aligned} \mathbf{v} &= 3t^2\mathbf{i} + 12t\mathbf{j} + 0\mathbf{k} \\ &= 3(2^2)\mathbf{i} + 12(2)\mathbf{j} + 0\mathbf{k} \\ &= 12\mathbf{i} + 24\mathbf{j} + 0\mathbf{k} \end{aligned}$$

The acceleration vector at $t = 2$

$$\begin{aligned} \mathbf{a} &= 6t\mathbf{i} + 12\mathbf{j} + 0\mathbf{k} \\ &= 6(2)\mathbf{i} + 12\mathbf{j} + 0\mathbf{k} \\ &= 12\mathbf{i} + 12\mathbf{j} + 0\mathbf{k} \end{aligned}$$

Calculating the magnitude

The position vector $\mathbf{r}$ m of a particle is given by

$$\mathbf{r} = 3t^2\mathbf{i} - (2t+1)\mathbf{j}$$

Find the magnitude of the position, velocity and acceleration vectors of the particle when $t = 3$s.

Differentiate the position vector to get the velocity and acceleration vectors:

$$\mathbf{v} = 6t\mathbf{i} - 2\mathbf{j} \quad \text{and} \quad \mathbf{a} = 6\mathbf{i} - 0\mathbf{j}$$

Substitute the value $t = 3$ into each vector to obtain its value at the required time and use Pythagoras' Theorem to calculate the magnitude:

$$\mathbf{r} = 27\mathbf{i} - 7\mathbf{j} \implies r = \sqrt{27^2 + (-7)^2} = \sqrt{778} \approx 27.9 \text{ m}$$

$$\mathbf{v} = 18\mathbf{i} - 2\mathbf{j} \implies v = \sqrt{18^2 + (-2)^2} = \sqrt{328} \approx 18.1 \text{ m s}^{-1}$$

$$\mathbf{a} = 6\mathbf{i} - 0\mathbf{j} \implies a = \sqrt{6^2 + 0^2} = \sqrt{36} = 6 \text{ m s}^{-2}$$

Since the acceleration vector does not depend on t we note that it must be constant in value.

Integrating the velocity or acceleration vectors

Since integration is the reverse of differentiation we find the velocity and distance from acceleration:

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \qquad \Rightarrow \mathbf{r} = \int \mathbf{v}\,\mathrm{d}t$$

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \qquad \Rightarrow \mathbf{v} = \int \mathbf{a}\,\mathrm{d}t$$

The integration is carried out on each component of the vector.

Integrating vectors will also introduce a constant vector of integration. As usual, boundary conditions must be applied to determine its value.

Integrating the velocity vector

A particle is moving in a plane in such a way that its velocity at time t is given by $(2t\mathbf{i}+3t^2\mathbf{j})\,\mathrm{m\,s^{-1}}$. Initially, the position vector of the particle, relative to the origin, is $(5\mathbf{i}-8\mathbf{j})$ m.
Find the position vector of the particle when $t=3$ seconds and the distance of the particle from the origin.

Find the position vector by integrating the velocity:

$$\begin{aligned}\mathbf{r} &= \int \mathbf{v}\,\mathrm{d}t \\ &= \int 2t\mathbf{i}+3t^2\mathbf{j}\,\mathrm{d}t \\ \mathbf{r} &= t^2\mathbf{i}+t^3\mathbf{j}+\mathbf{C}\end{aligned}$$

where $\mathbf{C}$ is a constant vector. Initially (when $t=0$) the position vector is $5\mathbf{i}-8\mathbf{j}$

$$5\mathbf{i}-8\mathbf{j}=0\mathbf{i}+0\mathbf{j}+\mathbf{C} \quad \Rightarrow \mathbf{C}=5\mathbf{i}-8\mathbf{j}$$

The position vector is

$$\begin{aligned}\mathbf{r} &= t^2\mathbf{i}+t^3\mathbf{j}+\mathbf{C} \\ &= t^2\mathbf{i}+t^3\mathbf{j}+5\mathbf{i}-8\mathbf{j} \\ \therefore \mathbf{r} &= (t^2+5)\mathbf{i}+(t^3-8)\mathbf{j}\end{aligned}$$

When $t=3$ the position vector is $\mathbf{r}=14\mathbf{i}+19\mathbf{j}$.

The distance from the origin is

$$r=|\mathbf{r}|=\sqrt{14^2+19^2}=\sqrt{557}$$

So $r\approx 23.6$ metres when $t=3$.

Integrating the acceleration vector

A particle with acceleration vector $\frac{32}{t^3}\mathbf{j}\,\mathrm{m\,s^{-2}}$ is moving such that when $t = 2$ seconds, its velocity vector is $(9\mathbf{i} - 4\mathbf{j})\,\mathrm{m\,s^{-1}}$ and its position vector is $(18\mathbf{i} + 8\mathbf{j})\,\mathrm{m}$.
(a) Find $\mathbf{v}$ and $\mathbf{r}$ in terms of t. (b) Find the value of t when $\mathbf{v}$ is perpendicular to $\mathbf{r}$

Part (a). Integrate the acceleration vector

$$\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int 32t^{-3}\mathbf{j}\,\mathrm{d}t$$
$$\mathbf{v} = -\tfrac{16}{t^2}\mathbf{j} + \mathbf{C}$$

where $\mathbf{C}$ is a constant vector. When $t = 2$ we know $\mathbf{v} = 9\mathbf{i} - 4\mathbf{j}$:

$$9\mathbf{i} - 4\mathbf{j} = -\tfrac{16}{2^2}\mathbf{j} + \mathbf{C} \quad \Rightarrow \mathbf{C} = 9\mathbf{i}$$

So the velocity vector is $\mathbf{v} = 9\mathbf{i} - \frac{16}{t^2}\mathbf{j}$.

Integrate again to get the position vector:

$$\mathbf{r} = \int 9\mathbf{i} - \tfrac{16}{t^2}\mathbf{j}\,\mathrm{d}t$$
$$= 9t\mathbf{i} + \tfrac{16}{t}\mathbf{j} + \mathbf{C_2}$$

where $\mathbf{C}_2$ is a constant vector.

We are given $t = 2$, $\mathbf{r} = 18\mathbf{i} + 8\mathbf{j}$. $\therefore \mathbf{C_2} = \mathbf{0}$.

The position vector is $\mathbf{r} = 9t\mathbf{i} + \frac{16}{t}\mathbf{j}$.

Part (b). The vectors are perpendicular when $\mathbf{r}.\mathbf{v} = 0$

$$\left(9t\mathbf{i} + \tfrac{16}{t}\mathbf{j}\right).\left(9\mathbf{i} - \tfrac{16}{t^2}\mathbf{j}\right) = 0$$
$$(9t)(9) + \left(\tfrac{16}{t}\right)\left(-\tfrac{16}{t^2}\right) = 0$$
$$81t - \tfrac{256}{t^3} = 0$$
$$81t^4 = 256$$
$$\therefore t = \tfrac{4}{3}$$

Test yourself...

Let's practice what we've learned with a quick test...

1. A particle moving along a line has a speed

$$v = 4\sin 2t$$

The particle is initially at $r = 0$. Find an expression for $r(t)$.

2. For the particle in (1) find the first time when $a = 4$
3. A particle has a velocity vector $\mathbf{v} = 3t^2\mathbf{i} + (t - 3)\mathbf{j}$. Find a when $t = 2$.
4. The particle in (3) is initially at $(-3, 4)$. Find an expression for $\mathbf{r}$.

Answers:

1. $r(t) = 2 - 2\cos 2t$
2. $a = 8\cos 2t$; therefore $a = 4$ when $t = \frac{1}{6}\pi$
3. $a = \sqrt{145}$.
4. $\mathbf{r} = (t^3 - 3)\mathbf{i} + (\frac{1}{2}t^2 - 3t + 4)\mathbf{j}$