# Circle Geometry

## Geometry 2

INU0114/514 (Maths 1)

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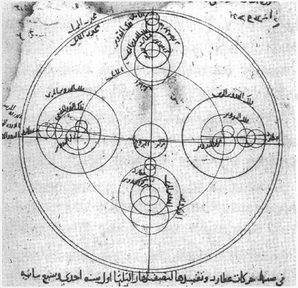
## Overview

We will extend our knowledge of geometry to problems connected to circles.

- Find the Cartesian equation of a circle of given centre and radius.
- Find the centre and radius of a circle from its Cartesian equation
- Solve problems with circles and straight lines
    - Determine whether a given point is inside, outside or on the circumference of a circle
    - Find the intersection points of a line with a circle
    - Find the equation of a tangent to a circle at a given point.

# History

The circle has had a special place in mathematics and science throughout history. It was often viewed as a perfect and divine shape.



Model of the orbit of Mercury (Ibn al-Shatir *c*14th century)

The shape of the Sun and Moon appeared to be perfect circles. Early models of the solar systems used circular orbits for the Sun and planets as they revolved around the Earth.

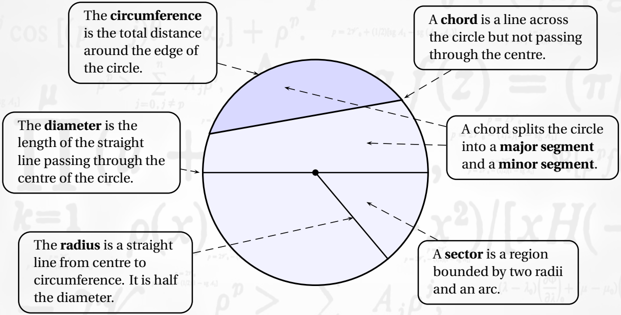
The truth turned out to be very different, with the Earth and other planets following ellipses around the Sun.

The circle is utilised in engineering in the form of wheels and gears.

For a student, the geometry of a circle introduces methods and techniques which can be applied to more complicated curves.

# Introduction

A circle is a set of points at a fixed distance (the radius) from a fixed point (the centre). The length of the radius is usually represented by $r$.



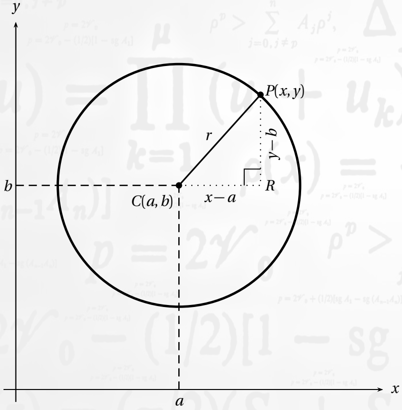

The ratio of the circumference to the diameter is a constant; is has the same value for all circles.It is denoted by $\pi$, where:

$$\pi = \frac{\text{Circumference}}{\text{Diameter}} = 3.14159265358979323846\ldots$$

The dots indicate that the decimal expansion continues forever.

# Cartesian equation of a circle

The picture shows a circle with centre $C$ at the point $(a, b)$, with radius $r$. Let $P$ be a point on the circumference with coordinates $(x, y)$.




Using Pythagoras's Theorem:

$$CP^2 = r^2$$
$$CR^2 + RP^2 = r^2$$

But $CR = x - a$ and $RP = y - b$ so:

$$\boxed{(x-a)^2 + (y-b)^2 = r^2}$$

## Centre and radius of a circle

Find the centre and radius of the circle with equation:

$$(x-6)^2+(y+3)^2=64$$

Comparison of each term with those in the general equation shows that $a=6$, $b=-3$ and $r^2=64$.

Therefore, the circle is centred at the point $(6,-3)$ and the radius is $r=\sqrt{64}=8$

We use the positive root because it represents length.

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Note that the circle equation can also be given in an alternate form by expanding the brackets and setting the RHS equal to zero.

$$\begin{aligned}(x-6)^2+(y+3)^2 &= 64\\ (x^2-12x+36)+(y^2+6y+9)-64 &= 0\\ x^2+y^2-12x+6y-19 &= 0\end{aligned}$$

# Cartesian equation (alternate form)

If an equation is not given in the form shown previously then we can convert it by completing the square on the $x$ and $y$ terms. This allows us to find the centre and radius easily.

## Centre and radius of a circle

A circle is given by the equation

$$x^2+y^2-6x+12y+20=0$$

Find the centre and radius of the circle.

First, it's helpful to regroup the terms:

$$\left[x^2-6x\right]+\left[y^2+12y\right]=-20$$

We carry out the "completing the square" process for the terms in brackets:

$$(x-3)^2-9+(y+6)^2-36=-20$$

Finally, simplify and take the constants to the RHS

$$(x-3)^2+(y+6)^2=25$$

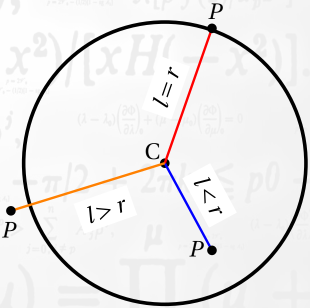
The circle has centre $(3,-6)$ and a radius is $\sqrt{25}=5$.

# Circles and points

Given a circle equation and a point, how can we tell (without a picture) whether the point is inside, outside circle or 'on' the edge of a circle?

For a circle centred at the point $C(a,b)$ and a point $P(x,y)$ the distance, $l$ between $P$ and $C$ is:

$$l = \sqrt{(x-a)^2 + (y-b)^2}$$




For the circle and the point $P$ we know that:

$l < r$ means $P$ is inside the circle.
$l > r$ means $P$ outside the circle.
$l = r$ means $P$ is on the circle.

## A circle and a point

Determine whether the point $(3,4)$ is located inside or outside the circle given by $(x-2)^2+(y+5)^2=20$.

Inspection of the circle equation reveals the centre of the circle is the point $(2,-5)$ and the radius is $r=\sqrt{20}$.

We must calculate the distance $d$ between the centre $(2,-5)$ and the given point $(3,4)$.

This is given by

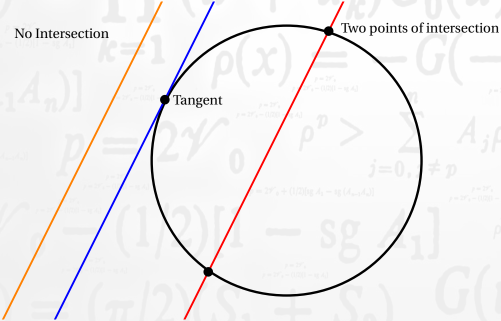
$$\begin{aligned} d &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ &= \sqrt{(3-2)^2+(4-(-5))^2} \\ &= \sqrt{1^2+9^2} \\ \therefore d &= \sqrt{82} \end{aligned}$$

Since $d>r$, the given point is outside the circle.

# Straight Lines and Circles

An interesting problem in geometry is to determine the points of intersection of a line with a circle. There are three possible cases to consider:

- The line intersects the circle at two points.
- The line touches the circle at one point, i.e. the line is a *tangent* to the circle.
- The line does not touch the circle.



## Line intersection with a circle

Find the coordinates of the points where the line $y = x + 1$ intersects with the circle $(x+2)^2 + \left(y-1\right)^2 = 4$.

To find an intersection we solve the equation of the circle and the line as simultaneously.

Substitute for $y$ in the equation of the circle:

$$\begin{aligned}(x+2)^2 + (y-1)^2 &= 4\\ (x+2)^2 + ((x+1-1)^2 &= 4\\ (x+2)^2 + x^2 &= 4\end{aligned}$$

Expand the brackets and simplify:

$$\begin{aligned}2x^2 + 4x + 4 &= 4\\ 2x^2 + 4x &= 0\\ 2x(x+2) &= 0\end{aligned}$$

Therefore $x = 0$ or $x = -2$.

Using $y = x + 1$ with each $x$:

When $x = -2$, $y = -1$

And $x = 0$, $y = 1$.

The two points of intersection are $(-2, -1)$ and $(0, 1)$.

## Line intersection with a circle?

Find the coordinates of the points where the line $y = x - 1$ intersects with the circle $(x+2)^2 + \left(y-1\right)^2 = 4$.

Substitute for $y$ in the circle equation; we get a quadratic in $x$.

$$\begin{aligned} (x+2)^2 + ((x-1)-1)^2 &= 4 \\ (x+2)^2 + (x-2)^2 &= 4 \end{aligned}$$

Expand the brackets and simplify:

$$\begin{aligned} x^2 + 4x + 4 + x^2 - 4x + 4 &= 4 \\ 2x^2 + 4 &= 0 \\ 2x^2 &= -4 \\ x^2 &= -2 \end{aligned}$$

There is no real solution to this and so no point of intersection between the line the circle.

# Circle, line and discriminant

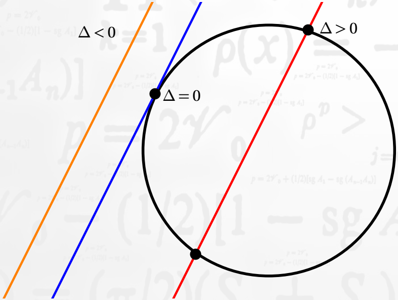
We've seen that solving to find the intersection between circle and line gives a quadratic equation.

We can use the discriminant to classify the intersection(s):

$\Delta > 0$ The line intersects the circle in two places.

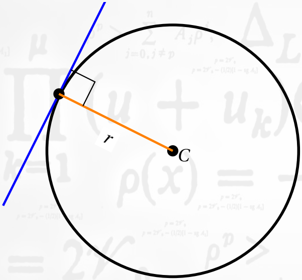
$\Delta = 0$ The line touches the circle in one place and is a tangent to the circle.

$\Delta < 0$ The line misses the circle. No Intersection.



# Tangent to a circle

Consider the tangent to a circle and the line from the centre of the circle to the point where the tangent touches the circle, i.e. the radius of the circle at that point.




The tangent to the circle and the radius to the point where the tangent touches the circle are *perpendicular* (90°) to each other. Therefore, if we multiply their gradients together we get a value of −1, i.e. $m_t m_r = -1$.

## Tangent to a circle

The point $P(2,3)$ is located on the circle $(x+1)^2+\left(y+1\right)^2=25$. Find the equation of the tangent at this point.

The centre of the circle $C$ is located at point $(-1,-1)$. Gradient of the radius $(m_r)$ from $C$ to $P$:

$$m_r = \frac{y_2-y_1}{x_2-x_1} = \frac{3-(-1)}{2-(-1)} = \frac{4}{3}$$

Since $m_t m_r = -1$ the gradient of the tangent $m_t = -\frac{3}{4}$. So the equation of the tangent has the form

$$y = -\frac{3}{4}x + c$$

The line passes through the point $(2,3)$ so

$$3 = -\frac{3}{4}(2)+c \quad \therefore c = \frac{9}{2}$$

So the tangent equation to the circle at $P$ is

$$y = -\frac{3}{4}x + \frac{9}{2}$$

# Test yourself...

You should be able to solve the following problems if you have understood everything in these notes.

1. Write down the equation for a circle with centre at $(-3, 6)$ and radius $\sqrt{5}$.
2. A circle is given by $x^2 + y^2 - 4x + 4y + 1 = 0$. Find the centre and radius.
3. Find the where the line $y = 2 - x$ intersects the circle $x^2 + y^2 - 4x - 14 = 0$.
4. Show that the line $3x - 4y - 3 = 0$ is a tangent to the circle $(x+6)^2 + (y-1)^2 = 25$.

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Answers:

1. $(x+3)^2 + (y-6)^2 = 5$
2. $(2, -2)$ and radius $\sqrt{7}$
3. $(-1, 3)$ and $(5, -3)$
4. Just need to show there is one point of contact; at $(-3, -3)$.