Straight line geometry

Geometry 1

INU0114/514 (Maths 1)

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Objectives

Geometry is the study of the relationships between points and lines. In this section we will restrict the maths to 2-dimensional (2D) cases. We'll study the geometry of lines in 3-dimensions in Semester 2.

- Calculate length and midpoint of a line segment.
- Definition of gradient (slope) of a straight line.
- Straight line equations:
    - Find the equation of a line through two given points
    - Find the equation of a line through a point with a given gradient
- Sketch graphs of linear functions.

History

The geometry taught in schools across the world comes from the work of the Greek mathematician Euclid of Alexandria (picture) who lived around 300BCE.

Euclid collected all of the mathematical knowledge of his age and compiled a book called "Elements".

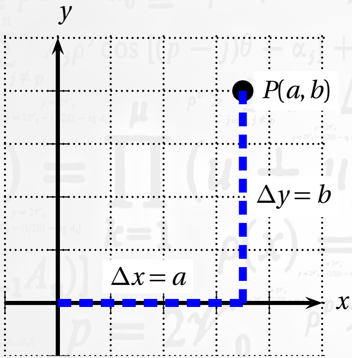
These books begin with a small list of axioms and statements and he used deduction to prove more advanced statements about geometry.

Algebra (developed much later in the Middle East) and geometry were connected by French mathematician René Descartes (picture) in the 17th century.

Descartes, known by his latin name *Cartesius*, used graphs to represent equations.

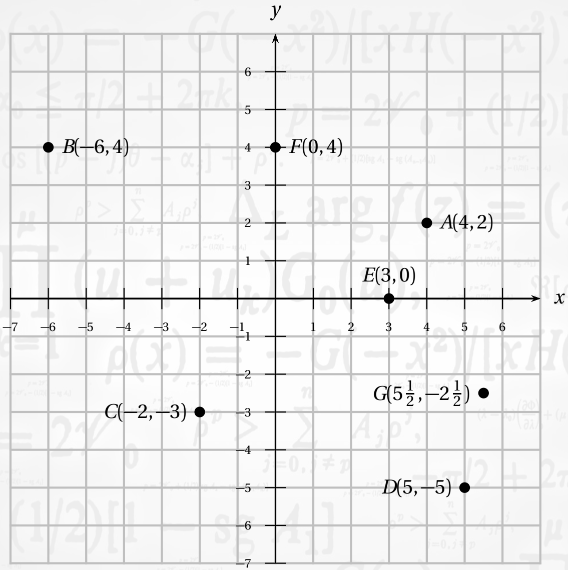
It meant problems could be converted from geometry to algebra (or vice-versa) for solution.

Cartesian Coordinates

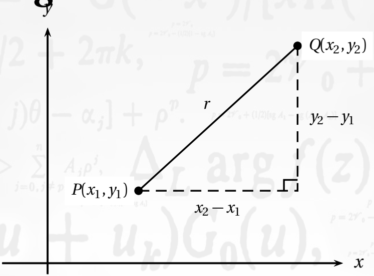


In Cartesian notation, the position of a point on the plane is measured with respect to a reference point O called the *origin*.
Every point on the plane can be assigned a pair of numbers called *coordinates*.

So if a point P is reached by starting at O and by moving a units in the direction of the horizontal x-axis and b units in the direction of the vertical y-axis then we would give the position of the point as (a, b).


y
x
B(−6, 4)
F(0, 4)
A(4, 2)
E(3, 0)
G(5 1/2, −2 1/2)
C(−2, −3)
D(5, −5)
−7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4 −5 −6 −7

Length of a line segment



The points P and Q are separated by a line of length r.The length of r can be calculated using Pythagoras's Theorem:

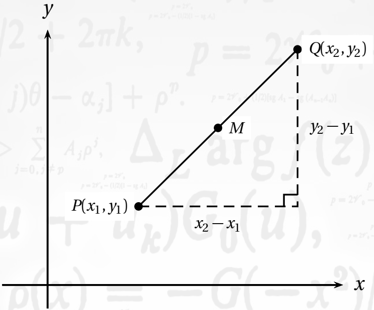
$$r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Therefore

$$\boxed{r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

We define r to be positive because it represents a length/distance.

Midpoint of a line segment



The midpoint M of the line segment is found by taking the mean (average) of the respective x and y values.

$$\text{Midpoint of } PQ = \left(\tfrac{1}{2}(x_1 + x_2),\ \tfrac{1}{2}(y_1 + y_2)\right)$$

Length of a line segment

Find the distance between the points $A(4,-2)$ and $B(9,-14)$.

Find the difference between each x and y coordinate (remember to subtract them in the same order!)

$$\begin{aligned} r &= \sqrt{(9-4)^2+(-14-(-2))^2} \\ &= \sqrt{5^2+(-12)^2} \\ &= \sqrt{25+144} \\ r &= \sqrt{169} \end{aligned}$$

The length of the line segment AB is 13 units.

Midpoint of a line segment

Find the midpoint of the line segment $A(12,6)$ and $B(6,-4)$.

Find the average of each x and y coordinate

$$\begin{aligned} x_m &= \tfrac{1}{2}(12+6)=9 \\ y_m &= \tfrac{1}{2}(6-4)=1 \end{aligned}$$

The midpoint of AB is the point $(9,1)$.

Test yourself...

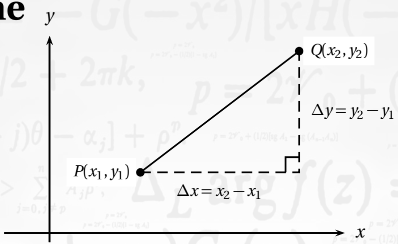
You should be able to solve the following problems based on the material covered so far.

1. Given $A(2,1)$ and $B(5,5)$ find the length of AB
2. Given $P(1,1)$ and $Q(3,3)$ find the length of PQ
3. Calculate the distance between $(-5,0)$ and $(1,-9)$.
4. Given $A(3,8)$ and $B(7,18)$ find the midpoint of AB
5. Given $P(1,-16)$ and $Q(-10,12)$ find the midpoint of PQ

Answers:

1. Length is 5
2. Length is $\sqrt{8}$ (or $2\sqrt{2}$)
3. Length is $\sqrt{117}$
4. Midpoint is $(5,13)$
5. Midpoint is $(-\frac{9}{2},-2)$ or $(-4.5,-2)$

Gradient of a line



The gradient m of the line is defined using changes in x and y directions by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

This definition leads to the following terminology:

Positive gradient y increases as x increases. The line looks like /

Negative gradient y dereases as x increases. The line looks like \

Zero gradient when $\Delta y = 0$ then the line is horizontal like —

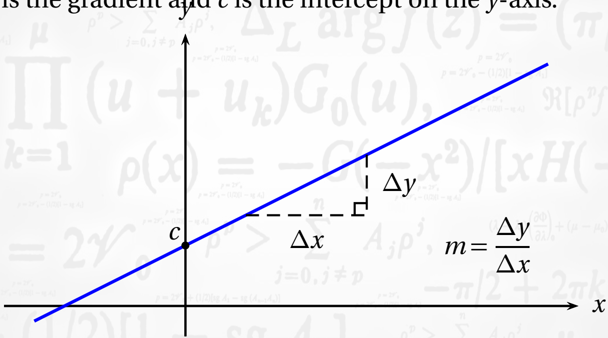
Undefined gradient when $\Delta x = 0$ then the line vertical like |

Straight line equations and graphs

A straight line equation has the general form

$$y = mx + c$$

where m is the gradient and c is the intercept on the y-axis.

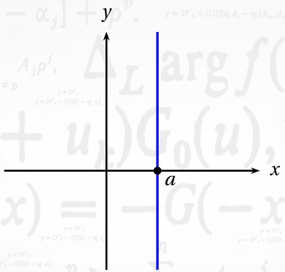


Interactive version:
http://www.geogebratube.org/student/m43224

Vertical straight line (undefined gradient)

The equation $y = mx + c$ doesn't apply to every straight line.

The picture below shows a vertical line:



All the points on this line share the same x-value of a.

We'd give the line equation as

$$x = a$$

in this case.

Line equation: alternate form

The equation of a straight line may also we written in the form

$$ax + by + c = 0$$

where a, b and c are integers.

To find the gradient and y-intercept we need to rearrange and write it in the form

$$y = mx + c$$

Note that the c in $ax + by + c = 0$ is different from the c in $y = mx + c$.

Changing the form

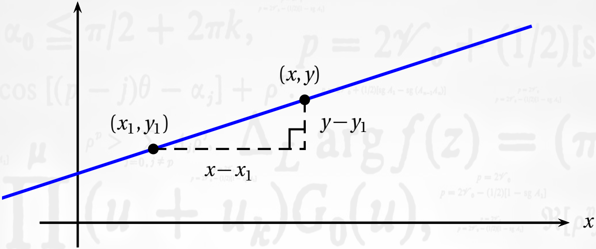
Find the gradient and y-intercept of the line $3x - 4y + 12 = 0$.

Rearrange to make y the subject: $y = \frac{3}{4}x + 3$

Compare this the general equation $[y = mx + c]$ we see $m = \frac{3}{4}$ and $c = 3$.

So the gradient is $\frac{3}{4}$ and the y-intercept is $(0, 3)$.

Line equation: given a point and gradient



Given that a straight line passes through a given point (x_1, y_1) then the relationship to any other point (x, y) on the line is found from

$$m = \frac{y - y_1}{x - x_1}$$

Rearrange to get

$$y - y_1 = m(x - x_1)$$

Given a point on the line (x_1, y_1) and the gradient m we can use this to find the equation of the line.

Line equation: given the gradient and a point on the line

Find the equation of the line with gradient -5 passing through the point $(1,2)$.

The line equation can be found from

$$y-y_1=m(x-x_1)$$

Substituting the gradient $m=-5$ and the point $x=1$, $y=2$ gives:

$$\begin{aligned} y-2 &= -5(x-1) \\ y-2 &= -5x+5 \\ y &= -5x+5+2 \\ y &= -5x+7 \end{aligned}$$

The line has equation $y=7-5x$.

Line equation: given two points

We previously saw that

$$y - y_1 = m(x - x_1)$$

If the gradient is not known then two points must be given to calculate it.

Given two points (x_1, y_1) and (x_2, y_2) then gradient of the line connecting them is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Line equation: given two points on the line

Find the equation of the line passing through the points $(-6, 10)$ and $(8, -18)$. Express the line in the form $ax + by + c = 0$, where a, b and c are integers.

First, find the gradient

$$m = \frac{-18 - 10}{8 - (-6)} = \frac{-28}{14} = -2$$

The line equation can be found from

$$y - y_1 = m(x - x_1)$$

Substituting the gradient $m = -2$ and one of the points (e.g. $x = -6$, $y = 10$) gives:

$$\begin{aligned} y - 10 &= -2(x + 6) \\ y - 10 &= -2x - 12 \\ y &= -2x - 12 + 10 \\ y &= -2x - 2 \end{aligned}$$

The line has equation $2x + y + 2 = 0$.

Sketching straight line graphs

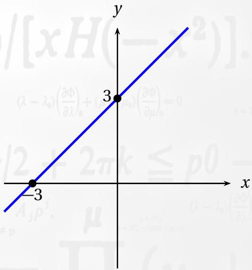
Since a straight line depends only on the values of m (gradient) and c (the y-intercept) in equation then it can be sketched without plotting a table of values. We just need two points to fix the position of the line on the graph.

The easiest method is the $x=0$, $y=0$ method.

Sketching a straight line graph

Sketch the graph of the line given by $y=x+3$.

- Begin with axes. Don't label with values yet!
- Substitute $x=0$. We get $y=3$ (y-intercept)
- Substitute $y=0$ and rearrange to get $x=-3$ (x-intercept).
- Draw the straight line through the two points.



Test yourself...

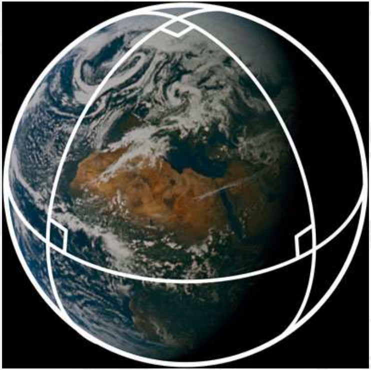
You should be able to solve the following problems based on the material covered so far.

1. Find the gradient of the line segment connnecting $A(2,1)$ and $B(5,5)$
2. Given that $y = 5 - 2x$, what is gradient of the line? Where is the y-intercept?
3. Given that $3x - 4y + 12 = 0$, what is gradient of the line? Where is the y-intercept?
4. Find the equation of the line with gradient 1 passing through the points $(5, -3)$.
5. Find the equation of the line passing through the points $(4,2)$ and $(-1,12)$.

Answers:

1. Gradient is $\frac{4}{3}$
2. Gradient is -2, y-intercept is 5
3. Gradient is $\frac{3}{4}$, y-intercept is 3
4. $y = x - 8$
5. $y = 10 - 2x$

Beyond flat space...



The rules and equations studied on this course were published more than 2000 years ago and we call them *Euclidean Geometry*. It is the geometry of flat surfaces.

Many of these rules and formulae are *not true* for lines on other types of surface — for example, the surface of a sphere (like the Earth!)

On a sphere, parallel lines *can* intersect (longitude lines intersect at the poles) and angles in a triangle *can* add to more than 180°.

Mathematicians have developed rules for other, non-Euclidean, geometries to better understand things like shape of the universe.

A branch of mathematics called *topology* generalises Euclid's rules to other types of space.