SKETCHING POLAR GRAPHS

GEOMETRY 3

INU0114/514 (MATHS 1)

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Introduction

We've already seen how to plot points in polar coordinates to graph a polar equation.

Do you recall how many points we needed to plot the equation $r = 3\sin 2\theta$? (A lot!)

In this presentation we'll see an alternative method for sketching polar curves. To graph the function

$$r = f(\theta)$$

first we'll sketch the graph $y = f(\theta)$ on Cartesian axes. This is usually much easier.

Then we'll use the Cartesian graph to construct the polar graph.

And we'll do all of this without polar graph paper to help!

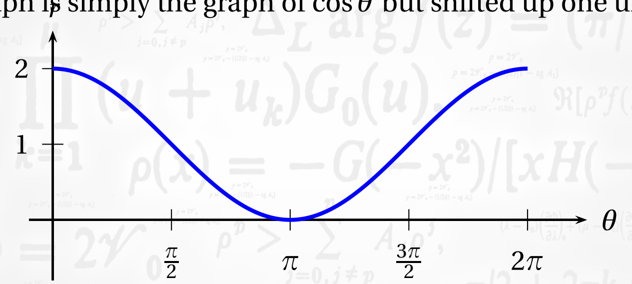
Let's go straight to some examples.

Sketching a polar curve

Sketch the curve $r = 1 + \cos\theta$, $0 \leq \theta \leq 2\pi$.

Begin by making a *Cartesian* sketch of r against θ.

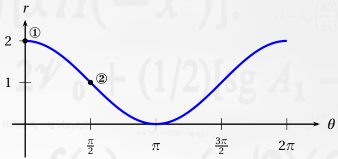
The graph is simply the graph of $\cos\theta$ but shifted up one unit.



We'll refer to this graph as a guide and build the polar graph in stages.

Look at the curve between the points denoted by ① and ②.

As the angle θ increases from 0 to $\frac{\pi}{2}$, then r decreases from 2 to 1.



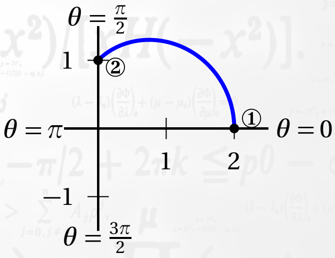
We'll transfer this information polar axes.

Start at the point $(2;0)$ and end at $(1;\frac{\pi}{2})$.

Draw a curve which turns through the angle from 0 to $\frac{\pi}{2}$.

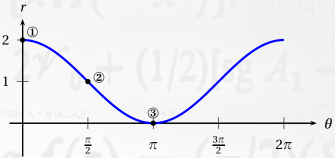
While we draw the curve we take care to decrease the distance from the origin from 2 to 1.

Notice that $r > 1$ for all points on this part of the curve.



Now, look at the curve between the points ② and ③.

As angle θ increases from $\frac{\pi}{2}$ to π, then r decreases from 1 to 0.

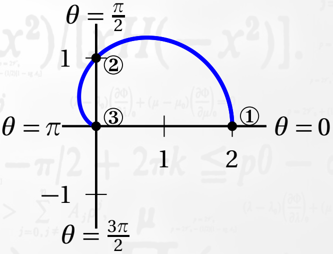


Again we transfer this behaviour to the polar axes.

Continuing from $(1; \frac{\pi}{2})$ and end at $(0; \pi)$.

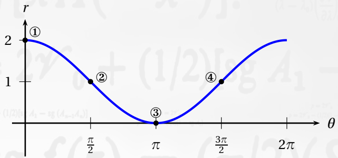
Draw a curve which turns through the angle from $\frac{\pi}{2}$ to π.

The curve is dragged towards the origin as the angle approaches π.



Examine the guide graph between ③ and ④.

As θ increases from π to $\frac{3\pi}{2}$ then r increases from 0 to 1.

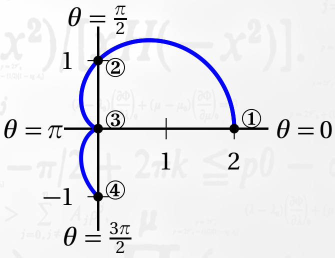


Go to the polar axes and sketch the next section of the curve.

Continue from $(0;\pi)$ and end at $(1;\frac{3\pi}{2})$.

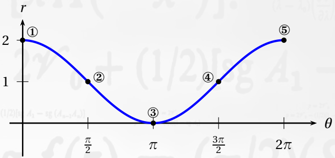
Draw a curve which turns through the angle from π to $\frac{3\pi}{2}$.

The curve is pulled outwards from the origin as the angle approaches $\frac{3\pi}{2}$.



The last section of the graph between ④ and ⑤.

As θ increases from $\frac{3\pi}{2}$ to 2π then r increases from 1 to 2.

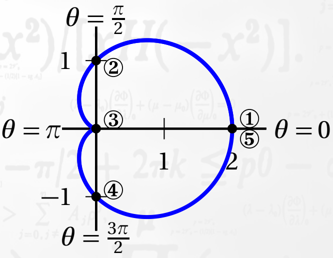


Go to the polar axes and sketch the final part of the curve.

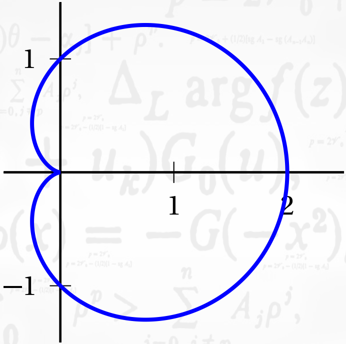
Continue from $(1; \frac{3\pi}{2})$ and end at $(2; 2\pi)$.

Draw a curve which turns through the angle from $\frac{3\pi}{2}$ to 2π.

The curve is pulled outwards from 1 to 2 as the angle increases to 2π.



Here is the final version of the graph $r = 1 + \cos\theta$, $0 \le \theta \le 2\pi$.



This is a cardioid (heart-shaped) curve.

Some advice: angle increments

In the first example we plotted a series of points at intervals of $\frac{\pi}{2}$ and then inferred the behaviour of the curve between those points.

Generally, when plotting polar graphs we want to know which angles are valuable for decoding the shape of the of the curve.

A useful strategy for an equation with a multiple angle, e.g.,

$$r = 3\sin 2\theta \qquad r = 4\cos 3\theta$$

is to set $n\theta = \frac{\pi}{2}$.

For the two examples above: we would look for angle increments of $\frac{\pi}{4}$ for the first, and $\frac{\pi}{6}$ for the second.

We'll see how this works on the next example.

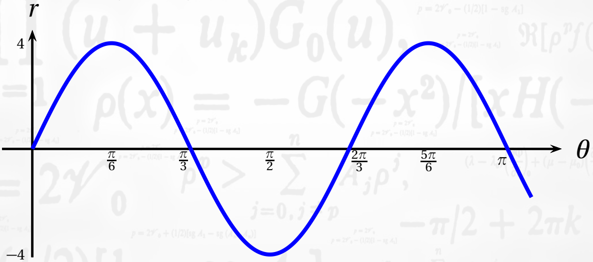
Polar graph sketch

Sketch the graph of the function $r = 4\sin 3\theta$.

The Cartesian graph of the function can obtained from the sine curve: stretch vertically by a factor of 4 and the period decreased by a factor of 3.

You might want to take this opportunity to revise trig transformations!

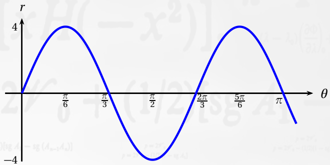
Here is the Cartesian guide graph:



The useful angles to consider on the polar graph are those where $3\theta = \frac{\pi}{2}$, in other words, increments of $\frac{\pi}{6}$.

We're going to build up the polar sketch by examining each section of the Cartesian graph.

Here is the Cartesian graph again for easy reference!



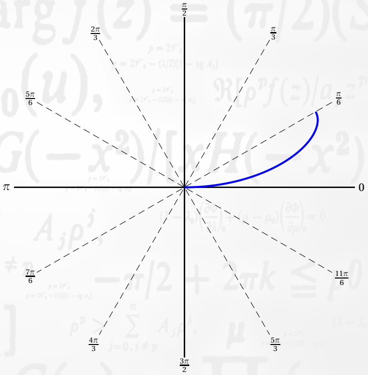
Let's draw the polar axes.

Mark lines at angle increments of $\frac{\pi}{6}$.

Guide graph: between 0 and $\frac{\pi}{6}$: r increases from 0 to 4.

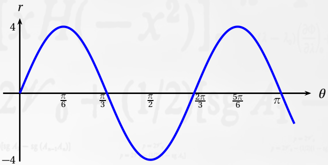
Polar graph: start at $(0; 0)$ and end at $(4; \frac{\pi}{6})$.

Draw the first section of the curve as shown.



We're going to build up the polar sketch by examining each section of the Cartesian graph.

Here is the Cartesian graph again for easy reference!

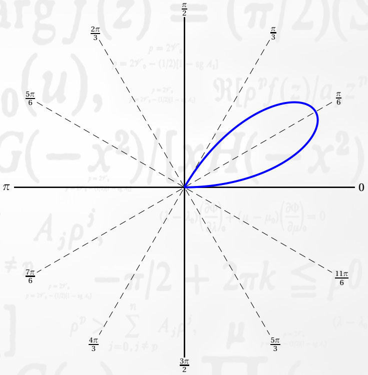


Let's take a look at the next section.

Guide graph: between $\frac{\pi}{6}$ and $\frac{\pi}{3}$: r decreases from 4 to 0.

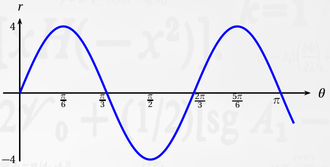
Polar graph: start at $(4; \frac{\pi}{6})$ and end at $(0; 0)$.

The symmetry of the sine curve suggests this is going to be a reflection of the first curve we drew!



We're going to build up the polar sketch by examining each section of the Cartesian graph.

Here is the Cartesian graph again for easy reference!



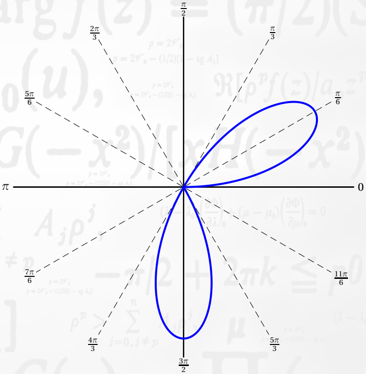
On the next part of the guide graph, r becomes negative — we plot directly away from that angle. Add π!

Guide graph: $\frac{\pi}{3}$ to $\frac{\pi}{2}$: r goes $0 \to -4$.

Polar graph: start $(0; \frac{\pi}{3})$ and end $(-4; \frac{\pi}{2})$.

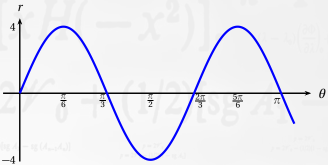
So: curve goes from origin to $(4; \frac{3\pi}{2})$.

Then the guide graph goes to zero between $\frac{\pi}{2}$ and $\frac{2\pi}{3}$; the polar graph decreases from 4 to zero between $\frac{3\pi}{2}$ and $\frac{5\pi}{3}$.



We're going to build up the polar sketch by examining each section of the Cartesian graph.

Here is the Cartesian graph again for easy reference!



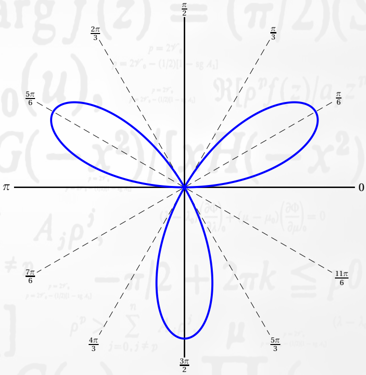
On this section of the guide graph, r is positive again.

Guide graph: $\frac{2\pi}{3}$ to $\frac{5\pi}{6}$: r goes $0 \longrightarrow 4$.

Polar graph: start $(0; \frac{2\pi}{3})$ and end $(4; \frac{5\pi}{6})$.

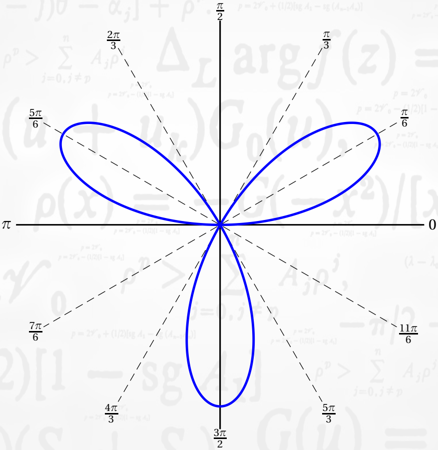
The guide graph shows r going $4 \longrightarrow 0$ in the final interval.

The polar graph goes $4 \longrightarrow 0$ between $\frac{5\pi}{6}$ and π.



If we try to continue following the guide graph for angles larger than π we find the polar curve begins to repeat and we simply retrace the sections already drawn.

Here is the final curve for $r = 4\sin 3\theta$, $0 \le \theta \le \pi$.



Test yourself

If you've read and understood the examples in these notes, you should be able to draw the following polar curves.

Consider the functions

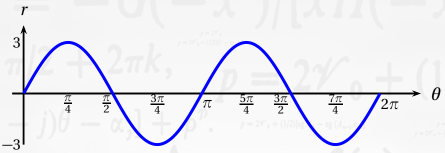
1. $r = 3\sin 2\theta$
2. $r = 1 + 2\cos\theta$

In each case

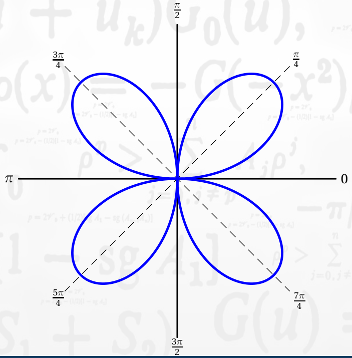
- Sketch the Cartesian guide graph.
- Use the guide graph to plot the polar graph.

The answers are shown on the next two slides.

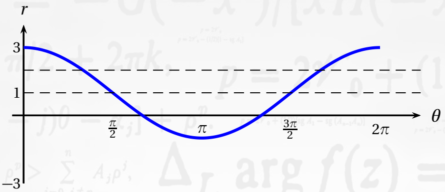
Guide graph for $r = 3\sin 2\theta$.



We need to put angle increments of $2\theta = \frac{\pi}{2}$ (so $\theta = \frac{\pi}{4}$) on the polar graph.



Guide graph for $r = 1 + 2\cos\theta$. Dashed lines to show r values more clearly.



In this case angle increments of $\theta = \frac{\pi}{2}$ will be fine on the polar graph.