POLAR GEOMETRY

GEOMETRY 3

INU0114/514 (MATHS 1)

Dr Adrian Jannetta MIMA CMath FRAS

Objectives

In this short presentation we'll cover the following:

- Solving geometry problems in polar coordinates.
- Further practice of plotting polar coordinates
- Practice converting equations between polar and cartesian form.

Polar and Cartesian forms

The polar curve C has equation

$$r = 2(\cos\theta - \sin\theta)$$

Prove that this equation represents a circle and find the coordinates of the centre and the radius.

Let's convert to Cartesian form using $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$

$$r = 2\left(\frac{x}{r} - \frac{y}{r}\right)$$

Cross multiply by r $\quad r^2 = 2(x-y)$

Now change r^2: $\quad x^2 + y^2 = 2(x-y)$

Rearrange to standard form for a circle:

$$x^2 + y^2 - 2x + 2y = 0$$

Change to completed square form:

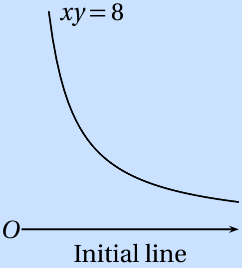
$$(x-1)^2 + (y+1)^2 = 2$$

Therefore: centre at $(1,-1)$ and radius is $\sqrt{2}$.

Geometry problems

Intersection of curves

The picture shows the curve $xy = 8$.



(a) Show that polar equation is $r^2 = 16\,\text{cosec}\,2\theta$.

(b) The circle whose polar equation is $r = 4\sqrt{2}$ intersects the curve at the points P and Q. Find the polar coordinates of P and Q.

(a) Given that

$$xy = 8$$

Use the transformation equations:

$$\begin{aligned}(r\cos\theta)(r\sin\theta) &= 8\\ r^2\sin\theta\cos\theta &= 8\end{aligned}$$

Rearrange to get

$$r^2 = \frac{8}{\sin\theta\cos\theta}$$

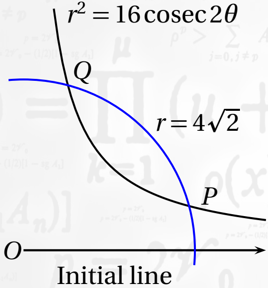
Next, use the trig identity $\sin 2\theta = 2\sin\theta\cos\theta$:

$$r^2 = \frac{8}{\frac{1}{2}\sin 2\theta} = \frac{16}{\sin 2\theta}$$

And we can rewrite this using the reciprocal function:

$$r^2 = 16\,\mathrm{cosec}\,2\theta$$

(b) Let's add the circle to the picture; it is centred at the origin and will intersect the other curve at two places.



To find the intersection — we must equate the line equations.

The circle equation is $r = 4\sqrt{2}$ and so $r^2 = 32$.

The intersection points satisfy

$$16\,\text{cosec}\,2\theta = 32 \qquad \therefore \text{cosec}\,2\theta = 2$$

We solve this by changing to the equivalent

$$\sin 2\theta = \tfrac{1}{2}$$

The principal value leads to

$$2\theta = 30^\circ \qquad \therefore \theta = 15^\circ$$

and the secondary value:

$$2\theta = 150^\circ \qquad \therefore \theta = 75^\circ$$

The points are

$$P(4\sqrt{2}; 15^\circ) \text{ and } Q(4\sqrt{2}; 75^\circ)$$

Intersection of two curves

A polar curve C is described by

$$r = \frac{1}{1-\cos\theta}$$

(a) Show that the Cartesian equation can be written as $y^2 = 2x+1$.
(b) Another curve is defined by $r = 13$ and intersects C at two places. Find the Cartesian coordinates of the intersection points.

(a) Substitute $\cos\theta = x/r$:

$$r = \frac{1}{1-\frac{x}{r}} = \frac{r}{r-x}$$

Assuming $r \neq 0$ we can simplify this to $$1 = \frac{1}{r-x}$$

Rearrange to get $$r - x = 1 \quad \therefore r = x+1$$

Square both sides:

$$\begin{aligned} r^2 &= (x+1)^2 \\ x^2+y^2 &= x^2+2x+1 \\ \therefore y^2 &= 2x+1 \end{aligned}$$

(b) The curve $r = 13$ is equivalent to $r^2 = 169$ and so the Cartesian equation is

$$x^2 + y^2 = 169 \tag{1}$$

The equation for C was

$$y^2 = 2x + 1 \tag{2}$$

Substitute (2) into (1):

$$\begin{aligned} x^2 + 2x + 1 &= 169 \\ x^2 + 2x - 168 &= 0 \end{aligned}$$

Solve to get $x = 12$ and $x = -14$.

Use equation (2) with these values.

$x = 12$ leads to $y^2 = 25$ and so $y = \pm 5$.

The intersection points are

$$(12, 5) \text{ and } (12, -5)$$

The value $x = -14$ does not give real values for y; (try it!)

Some advice...

We usually have a choice about converting equations to Cartesian or to polar form. Your experience of past problems will eventually guide you in deciding the best way forward but don't be afraid to try an alternative if the equations become too difficult; it might be easier if you change your approach.

For example — here is an alternative way to solve part (b) of the previous problem.

Curve C in polar form was

$$r = \frac{1}{1 - \cos\theta}$$

So the intersection with the other curve $r = 13$ can be found by equating them:

$$13 = \frac{1}{1 - \cos\theta} \qquad \Rightarrow \cos\theta = \frac{12}{13}$$

We know $x = r\cos\theta$ so

$$x = 13(\tfrac{12}{13}) = 12$$

Now we know $\cos\theta = \frac{12}{13}$ .... we can find $\sin\theta$ using the fundamental identity $\sin^2\theta \equiv 1 - \cos^2\theta$:

$$\sin^2\theta = 1 - (\tfrac{12}{13})^2 = \tfrac{144}{169}$$

Therefore

$$\sin\theta = \pm\frac{5}{13}$$

We know $y = r\sin\theta$ so

$$y = 13(\pm\tfrac{5}{13}) = \pm 5$$

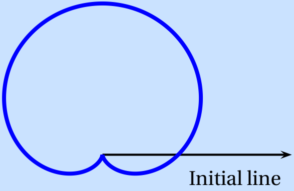
Since $x = 12$ then the intersection points are

$$(12, 5) \text{ and } (12, -5)$$

Was this an easier method? Perhaps....if your trig identity knowledge is good!

Intersection with a curve

Consider the cardioid curve C described by $r = 3(1+\sin\theta)$.



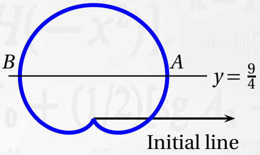
The curve is intersected by the line $y = \frac{9}{4}$ at two points A and B.

(a) Find the coordinates of A and B

(b) Calulate the length of AB.

(a) The line $y = \frac{9}{4}$ is a horizontal line.

This is shown in the picture; we'll call the intersection points A and B.



Let's change the line $y = \frac{9}{4}$ into polar form:

$$r \sin\theta = \frac{9}{4}$$

At the intersection we also know $r = 3(1 + \sin\theta)$:

$$3(1 + \sin\theta)\sin\theta = \frac{9}{4}$$

This is a quadratic in $\sin\theta$. We can rearrange to get this:

$$\begin{aligned} 4\sin^2\theta + 4\sin\theta - 3 &= 0 \\ (2\sin\theta - 1)(2\sin\theta + 3) &= 0 \end{aligned}$$

The root $\sin\theta = -\frac{3}{2}$ has no solutions for θ.

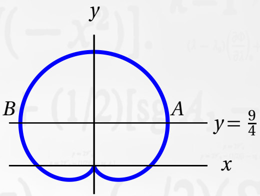
The other root is $\sin\theta = \frac{1}{2}$ and so $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$.

Using $r = 3(1 + \sin\theta)$ we calculate $r = \frac{9}{2}$ for both θ values. The intersections are

$$A\left(\frac{9}{2}, \frac{\pi}{6}\right) \quad \text{and } B\left(\frac{9}{2}, \frac{5\pi}{6}\right)$$

(b) The length AB is the distance between two points.

Let's find the cartesian coordinates...

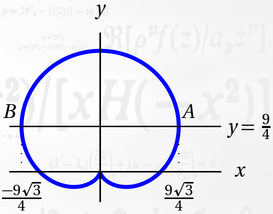


Since $y = \frac{9}{4}$ calculate the x-values with $x = r\cos\theta$.

We know $r = \frac{9}{2}$ and $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$.

The cartesian the coordinates are:

$$A\left(\frac{9\sqrt{3}}{4}, \frac{9}{4}\right) \quad \text{and } B\left(-\frac{9\sqrt{3}}{4}, \frac{9}{4}\right)$$



AB is the difference between the x-values on the line.

Therefore $AB = \frac{9\sqrt{3}}{4} - \left(-\frac{9\sqrt{3}}{4}\right) = \frac{9\sqrt{3}}{2}$