Polar equations and graphs

Geometry 3

INU0114/514 (Maths 1)

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Objectives

The purpose of this presentation is to cover the following topics:

- Polar coordinate system
- Conversion between polar and Cartesian coordinates
- Sketching polar functions
- Conversion between Polar equations and Cartesian equations
- Standard polar curves

Introduction

Polar coordinates are used to uniquely specify points in a plane.

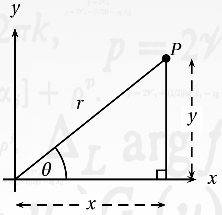
We are already used to Cartesian coordinates to specify our points on graphs but there are other ways of doing it.

In the polar coordinate system each point is specified by a distance r from the origin and an angle θ measured in an anticlockwise sense from the positive x-axis.

Thus we can specify the point P in the plane as $P(x, y)$ or $P(r; \theta)$ using Cartesian or polar coordinates respectively.

Note the convention of using a semi-colon (;) to separate the polar coordinates.

Coordinate system transformations



The following equations will transform polar coordinates to Cartesian coordinates:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{1}$$

To transform Cartesian coordinates to polar coordinates, we use Pythagoras's theorem:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{2}$$

where the angle θ must be given in the correct quadrant.

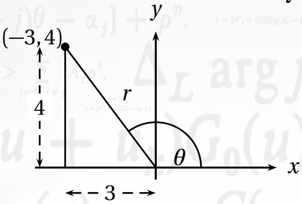
To uniquely specify each point on the plane, we must restrict the angle to the interval $0 \le \theta < 360°$ (or $0 \le \theta < 2\pi$).

This is identical to the method used for vector magnitude and direction.

Cartesian to polar conversion

Give the point $P(-3,4)$ in polar coordinates.

Draw a picture — it's a visual check that you're doing everything ok.



The $x = -3$ and $y = 4$ values give:

$$r^2 = (-3)^2 + 4^2 = 25 \quad \therefore r = 5$$

The principal value is $\theta = \tan^{-1}\left(\frac{4}{-3}\right) = -53.1°$.

The picture shows the point is in the second quadrant, so we add 180° to get $\theta = 126.9°$.

The polar coordinates are $(5; 126.9°)$.

Polar to Cartesian conversion

Give the point $Q(4;330°)$ in Cartesian coordinates.

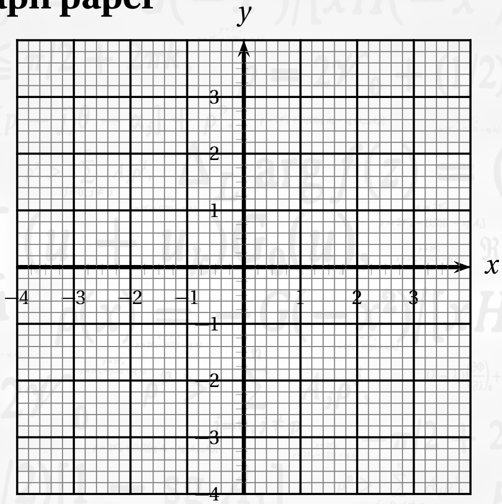
We are given $r = 4$ and $\theta = 330°$.

The coordinates are found using $x = r\cos\theta$ and $y = r\sin\theta$:

$$
\begin{aligned}
x &= 4\cos 330° = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3} \\
y &= 4\sin 330° = 4\left(-\frac{1}{2}\right) = -2
\end{aligned}
$$

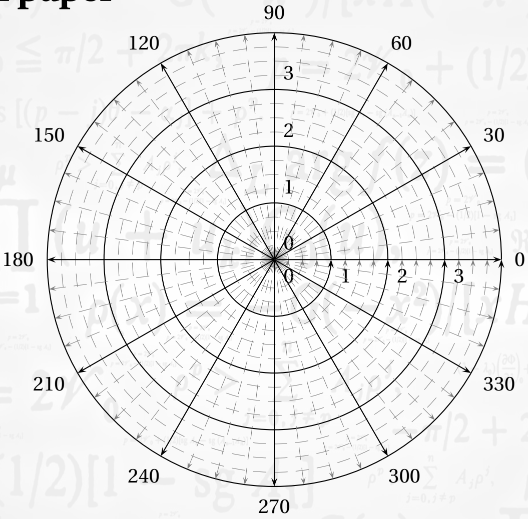
The Cartesian coordinates are $Q(2\sqrt{3}, -2)$.

Cartesian graph paper



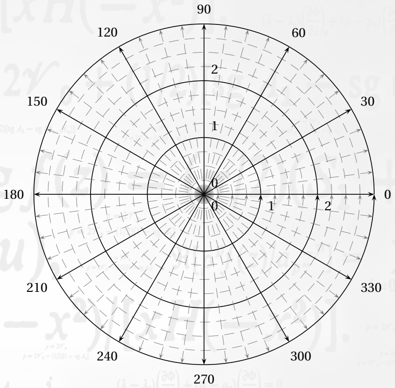
This is convenient for plotting Cartesian (x, y) points and curves.

Polar graph paper



This is convenient for plotting polar $(r;\theta)$ points and curves.

Plotting polar coordinates



Instructions for plotting the graph of $r = f(\theta)$.

Choose a value for θ and calculate r using $r = f(\theta)$. (Make a table of values).

Remember that θ is taken to be anti-clockwise from the positive horizontal direction.

We plot the point $(r; \theta)$ as follows:

- Measure a distance r from the origin in the direction of θ.
- Negative values of r are measured *away* from the direction of θ.

Plotting a polar curve

Plot the curve

$$r = 3\cos\theta \quad , 0 \le \theta \le 180^\circ$$

Let's make a simple table of values to help with the plotting.

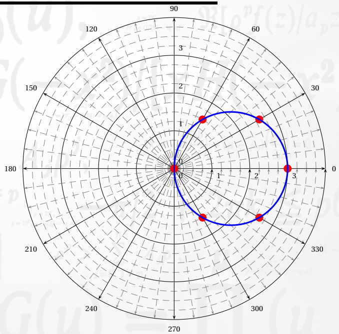
θ	0	30°	60°	90°	120°	150°	180°
r	3	2.6	1.5	0	−1.5	−2.6	−3

Let's plot the values.

Remember: negative r goes *away* from the angle!

When we're done with plotting points, we draw a smooth curve through them.

The curve is a circle; but that might not be obvious because we plotted only a few points. We'll prove it later.



Plotting a polar curve

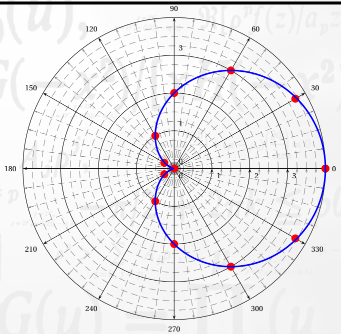
Plot the curve

$$r = 2 + 2\cos\theta \quad , 0 \leq \theta \leq 360^\circ$$

Let's make a simple table of values to help with the plotting.

θ	0	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
r	4	3.7	3	2	1	0.3	0	0.3	1	2	3	3.7	4

Let's plot the values.



When we're done with plotting points, we draw a smooth curve through them.

This heart-shaped curve is a known as a cardioid.

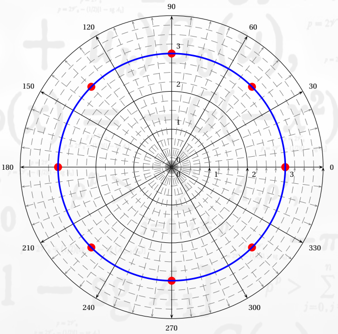
Plotting a polar curve

Plot the curve

$$r = 3 \quad , 0 \leq \theta \leq 360^\circ$$

We don't need a table of values for this. There is no θ in the polar equation. That means $r = 3$ for every value of θ!

For example, the points $(3; 0^\circ)$, $(3; 45^\circ)$, $(3; 90^\circ)$ ... etc., are all on the curve.



The equation $r = 3$ represents a circle (of radius 3, centred at the origin).

Plotting a polar curve

Plot the curve

$$r = 3\sin 2\theta \quad , 0 \leq \theta \leq 360^\circ$$

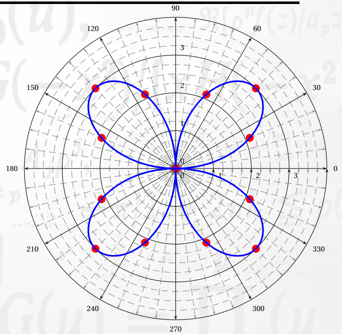
Begin, as usual, with a table of values to help with the plotting.

θ	0	45°	90°	135°	180°	225°	270°	315°	360°
r	0	3	0	-3	0	3	0	-3	0

In this case we probably need more points to see the shape of the curve clearly. Let's plot more points in between those given in the table (e.g. 22.5°, 67.5°).

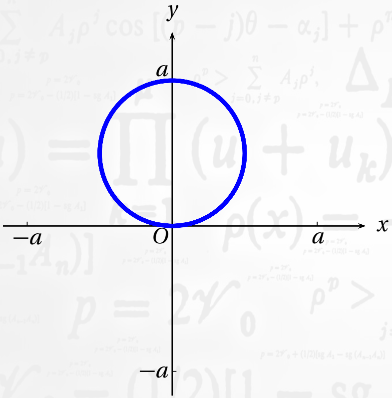
When we're done, we'll draw a smooth curve through the points.

In this example, plotting points to sketch the curve was not efficient. We'll see a better method later.

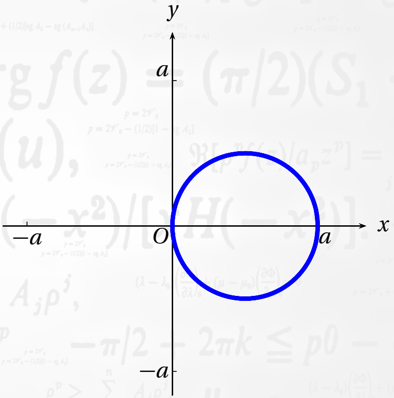


Standard Polar Curves

We'll finish this section by showing some well known polar curves.

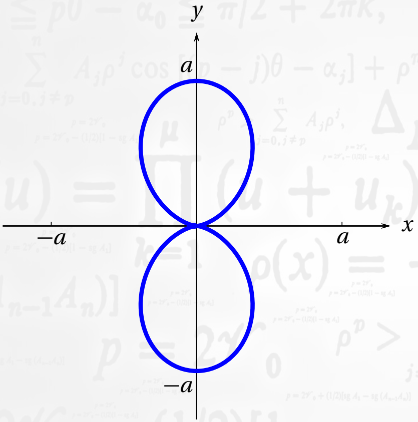


$r = a\sin\theta$

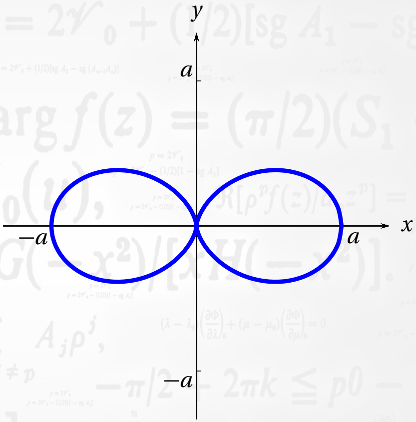


$r = a\cos\theta$

These curves are called *lemniscates.*

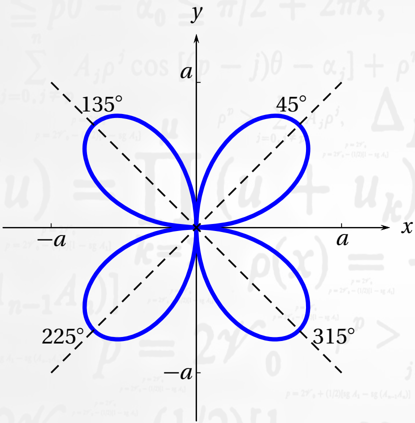


$r = a\sin^2\theta$

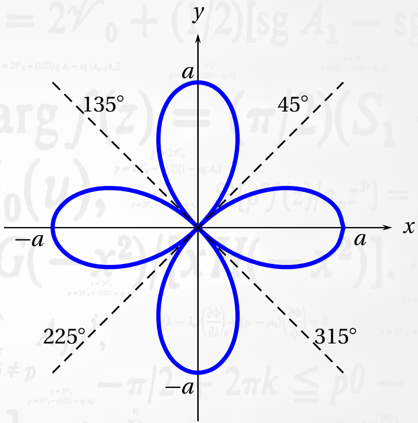


$r = a\cos^2\theta$

These curves are called *polar roses.*

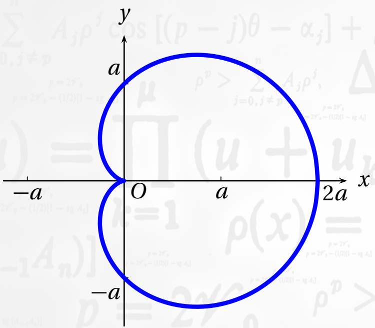


$r = a\sin 2\theta$



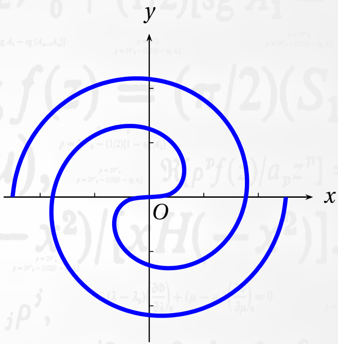
$r = a\cos 2\theta$

Cardioid (heart-shaped) curve



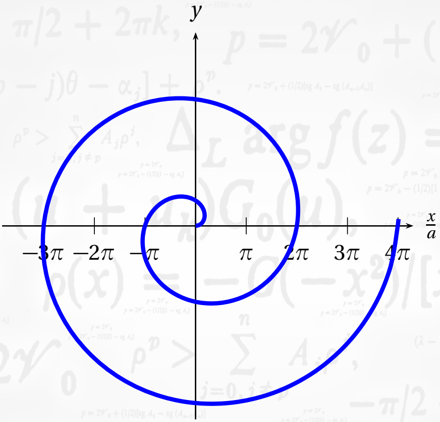
$$r = a(1+\cos\theta)$$

Fermat's Spiral



$$r^2 = a^2\theta$$

This last one is *Archimedes' Spiral.*



$$r = a\theta$$

Cartesian equations and polar equations

Earlier we saw how to convert coordinates of points between Cartesian and polar form. We need to learn to do the same for Cartesian equations and polar equations.

The same transformation equations apply but these slightly different rearrangements are often more useful.

$$x = r\cos\theta \qquad \therefore \cos\theta = \frac{x}{r}$$

$$y = r\sin\theta \qquad \therefore \sin\theta = \frac{y}{r}$$

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x}$$

Try to memorise these transformations. We will occasionally need to make use of trig identities – particularly the double angle formulae:

$$\sin 2\theta \equiv 2\sin\theta\cos\theta$$

$$\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$$

Polar equation to Cartesian form

Express $r = 3\cos\theta$ as a Cartesian equation.

Refer to the previous slide to see $\cos\theta = \dfrac{x}{r}$:

$$\begin{aligned} r &= 3\left(\frac{x}{r}\right) \\ r^2 &= 3x \end{aligned}$$

Also, $r^2 = x^2 + y^2$ so:

$$x^2 + y^2 = 3x$$

This is a Cartesian equation — the relationship is given in terms of x and y. Rearrange to show this a circle!

$$\begin{aligned} x^2 + y^2 - 3x &= 0 \\ (x - \tfrac{3}{2})^2 + y^2 &= \tfrac{9}{4} \end{aligned}$$

This circle is centred at $(\frac{3}{2}, 0)$ and has radius $r = \frac{3}{2}$. Compare this to the sketch you made earlier!

Polar equation to Cartesian form

Express $r^2 = 2\sin 2\theta$ as a Cartesian equation.

First, we need to express this in terms of $\sin\theta$ and $\cos\theta$:

$$r^2 = 2(2\sin\theta\cos\theta) = 4\sin\theta\cos\theta$$

We know $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$:

$$\begin{aligned} r^2 &= 4\left(\frac{x}{r}\right)\left(\frac{y}{r}\right) = \frac{4xy}{r^2} \\ r^4 &= 4xy \end{aligned}$$

Also, $r^2 = x^2 + y^2$ so:

$$(x^2+y^2)^2 = 4xy$$

This is a Cartesian equation — the relationship is given in terms of x and y.

Polar equation to Cartesian form

Express the polar equation $r = 2\tan\theta$ as a Cartesian equation.

We know $\tan\theta = \dfrac{y}{x}$:

$$r = \frac{2y}{x}$$

Also, $r = \sqrt{x^2+y^2}$ so:

$$\sqrt{x^2+y^2} = \frac{2y}{x}$$

This is a Cartesian equation. But perhaps this is a better way to write the function:

$$x^2+y^2 = \left(\frac{2y}{x}\right)^2 \quad \text{or} \quad x^2(x^2+y^2) = 4y^2$$

Cartesian equation to polar equation

Express the Cartesian equation $(x^2+y^2)^2 = 2x^2$ as a polar equation.

Given that $r^2 = x^2+y^2$ and $x = r\cos\theta$ so $(x^2+y^2)^2 = 2x^2$ becomes:

$$\begin{aligned} (r^2)^2 &= 2(r\cos\theta)^2 \\ r^4 &= 2r^2\cos^2\theta \\ r^2 &= 2\cos^2\theta \end{aligned}$$

Taking a square root would introduce $\pm$ into the equation so we'll just leave it in this form.

Cartesian equation to polar form

The Cartesian equation of an ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Write this as a polar equation in the form $r = f(\theta)$.

Given that $x = r\cos\theta$ and $y = r\sin\theta$ then

$$\begin{aligned}
\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} &= 1 \\
r^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) &= 1 \\
r^2\left(\frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2 b^2}\right) &= 1
\end{aligned}$$

This can be rearranged to get: $r^2 = \dfrac{a^2 b^2}{b^2\cos^2\theta + a^2\sin^2\theta}$

Take a square-root:: $r = \dfrac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$

Polar equations: angle measurement

So far, we have introduced the basic concepts of polar equations using angles measured in degrees.

Polar angles — be prepared to use radians!

Mathematicians can analyse polar curves with calculus or solve polar equations with numerical methods. In such cases it is necessary to use radians for angle measurements.

Test yourself

If you've read and understood the examples in these notes, you should be able to answer the following questions.

1. Write the coordinates $(-8, 0)$ in the polar form.
2. Express $(12; \frac{7\pi}{6})$ in as Cartesian coordinates.
3. Convert the polar equation $r^2 = 8\cos 2\theta$ to Cartesian form.
4. Write the equation $(x^2 + y^2)(x - 1)^2 = x^2$ in polar form.
5. Plot the graph of $r = 3\sin^2\theta$, $0 \leq \theta \leq 360°$

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1. $(8; 180°)$ or $(8; \pi)$.
2. $(-6\sqrt{3}, -6)$
3. $(x^2 + y^2)^2 = 8(x^2 - y^2)$ (hint: use $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$)
4. $r = \sec\theta + 1$

5.