Integration with trig identities

Calculus 9

INU0115/515 (Maths 2)

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Objectives

In this presentation we're going to tackle integration of trig functions. Before you begin:

- You should be comfortable doing integration by reverse chain rule.
- A list of common trig identities will also prove to be useful!

The strategy for the problems you're about to see is to replace the function with something we can integrate easily (such as a function on our list of standard integrals). There are many trig identities in mathematics but we'll begin with some simple examples to cover the most common cases you're likely to see.

Integrating $\tan^2 x$

Let's start with a simple example...

Using identities

Find $\int \tan^2 x \, \mathrm{d}x$

We can use the identity $\tan^2 x \equiv \sec^2 - 1$ in this case:

$$\int \tan^2 x \, \mathrm{d}x = \int (\sec^2 x - 1) \, \mathrm{d}x$$

The integral of $\sec^2 x$ is on our list of standard integrals:

$$\int \tan^2 x \, \mathrm{d}x = \int (\sec^2 x - 1) \, \mathrm{d}x = \tan x - x + C$$

Integrating $\sin^2 x$ and $\cos^2 x$

Double angle identities can be useful for integrating $\sin^2 x$ or $\cos^2 x$.

Using a double angle identity

Find $\int \sin^2 x \, dx$.

We can use the identity $\sin^2 \theta \equiv \frac{1}{2}(1-\cos 2\theta)$ to give us an expression that can be integrated more easily.

$$\int \sin^2 x \, dx = \frac{1}{2}\int (1-\cos 2x)\, dx$$

Using the reverse chain rule on the cosine term, we can write:

$$\begin{aligned} \int \sin^2 x \, dx &= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C \\ &= \frac{1}{4}(2x - \sin 2x) + C \end{aligned}$$

Higher powers of sine or cosine ($\sin^k x$ or $\cos^k x$, $k > 2$) will be examined later in the presentation but will not be assessed on the exam.

Double angle indentities

Integration with a double angle identity

Evaluate $\int_0^{\frac{\pi}{12}} 8\sin 2x\cos 2x\,\mathrm{d}x$

We'll need the identity $\sin 2\theta \equiv 2\sin\theta\cos\theta$. In this case, we have $\theta = 2x$ so that:

$$\begin{aligned} \sin 4x &\equiv 2\sin 2x\cos 2x \\ \therefore 4\sin 4x &\equiv 8\sin 2x\cos 2x \end{aligned}$$

This integral becomes:

$$\int_0^{\frac{\pi}{12}} 8\sin 2x\cos 2x\,\mathrm{d}x = \int_0^{\frac{\pi}{12}} 4\sin 4x\,\mathrm{d}x$$

Integrate with the reverse chain rule:

$$\begin{aligned} \int_0^{\frac{\pi}{12}} 8\sin 2x\cos 2x\,\mathrm{d}x &= [-\cos 4x]_0^{\pi/12} = \left(-\cos\tfrac{\pi}{3} - (-\cos 0)\right) = -\tfrac{1}{2} + 1 \\ \therefore \int_0^{\frac{\pi}{12}} 8\sin 2x\cos 2x\,\mathrm{d}x &= \tfrac{1}{2} \end{aligned}$$

Using compound angle identities

Recall the following identities for sine:

$$\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) \equiv \sin A \cos B - \cos A \sin B$$

and for cosine:

$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) \equiv \cos A \cos B + \sin A \sin B$$

We're going to need these for the next couple of examples.

Compound angle identities are useful for turning a product into a sum; if the integral is a sum of two terms then we can integrate each term without a problem.

Using compound angle identities

(a) Show that $\sin A \cos B \equiv \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
(b) Hence, evaluate the integral

$$\int_0^{\frac{\pi}{12}} \sin 6x \cos 3x \, dx$$

(a) Use the identies for $\sin(A+B)$ and $\sin(A-B)$

$$\begin{aligned} \sin(A+B) &\equiv \sin A \cos B + \sin B \cos A \\ \sin(A-B) &\equiv \sin A \cos B - \sin B \cos A \end{aligned}$$

Add the identities:

$$\sin(A+B) + \sin(A-B) \equiv 2 \sin A \cos B$$

Divide both sides by 2 to get:

$$\sin A \cos B \equiv \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

(b) To integrate the product $\sin 6x \cos 3x$ we'll use the identity obtained in part (a).

Set $A = 6x$ and $B = 3x$.

$$\begin{aligned}
\int_0^{\frac{\pi}{12}} \sin 6x \cos 3x \, \mathrm{d}x &\equiv \int_0^{\frac{\pi}{12}} \tfrac{1}{2}\left[\sin 9x + \sin 3x\right] \mathrm{d}x \\
&= \tfrac{1}{2}\left[-\tfrac{1}{9}\cos 9x - \tfrac{1}{3}\cos 3x\right]_0^{\frac{\pi}{12}} \\
&= -\tfrac{1}{18}\left[\cos 9x + 3\cos 3x\right]_0^{\frac{\pi}{12}} \\
&= -\tfrac{1}{18}\left[\left(\cos\tfrac{9\pi}{12} + 3\cos\tfrac{3\pi}{12}\right) - \left(\cos 0 + 3\cos 0\right)\right] \\
&= -\tfrac{1}{18}\left[\left(\cos\tfrac{3\pi}{4} + 3\cos\tfrac{\pi}{4}\right) - (1+3)\right] \\
&= -\tfrac{1}{18}\left[\left(-\tfrac{\sqrt{2}}{2} + \tfrac{3\sqrt{2}}{2}\right) - 4\right] \\
&= -\tfrac{1}{18}(\sqrt{2} - 4)
\end{aligned}$$

How can we figure out which compound angle identity to use?

Here's the list again:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \quad (1)$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B \quad (2)$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B \quad (3)$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B \quad (4)$$

If we have to integrate a product such as $\int \cos 6x \cos 2x \, dx$ then we'd use the last pair of identities. because they contain the product we need on the RHS.

Using compound angle identities

Find $\int \cos 6x \cos 2x \, dx$.

We can derive the identity we need from the cosine compound angle identities (equations 3 and 4):

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B$$

Therefore

$$\cos A\cos B=\tfrac{1}{2}[\cos(A+B)+\cos(A-B)]$$

That means we can express the integrand like this:

$$\begin{aligned}\int \cos 6x \cos 2x \, dx &= \tfrac{1}{2}\int [\cos 8x+\cos 4x] \, dx \\ &= \tfrac{1}{2}\left(\frac{\sin 8x}{8}+\frac{\sin 4x}{4}\right)+C \\ &= \tfrac{1}{16}(\sin 8x+2\sin 4x)+C\end{aligned}$$

Powers of sine, cosine and tangent (optional)

The following methods are optional and will not be assessed on the exam.

Trig identities are essential for integrating powers of sine, cosine or tangent. For example, we may wish to find $\int \cos^5 x \, dx$. The general procedure is decided by whether the power is odd or even (for sine and cosine) or a power of the tangent function.

1. Even powers of $\sin x$ and $\cos x$.
   Use double angle identities.
   1. Odd powers of $\sin x$ and $\cos x$.
      Use $\cos^2 x \equiv 1 - \sin^2 x$ or $\sin^2 x \equiv 1 - \cos^2 x$.
   2. Powers of $\tan x$.
      Use $\tan^2 x \equiv \sec^2 x - 1$.

We'll illustrate each of these cases with an example.

Even powers of sine and cosine

Find $\int \cos^4 x \, dx$

Separate the integrand into powers of $\cos^2 x$ and expand with the double angle identity:

$$\begin{aligned}\int \cos^4 x \, dx &= \int \left(\cos^2 x\right)^2 dx \\ &= \int \left[\tfrac{1}{2}(1+\cos 2x)\right]^2 dx \\ &= \tfrac{1}{4}\int (1+\cos 2x)^2 \, dx \\ &= \tfrac{1}{4}\int (1+2\cos 2x+\cos^2 2x) \, dx\end{aligned}$$

We must apply the double angle identity again to remove the remaining powers.

$$\int \cos^4 x \,\mathrm{d}x = \tfrac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,\mathrm{d}x$$

This time put $\cos^2 2x \equiv \frac{1}{2}(1+\cos 4x)$:

$$\begin{aligned} &= \tfrac{1}{4}\int \left(1 + 2\cos 2x + \tfrac{1}{2}(1+\cos 4x)\right)\mathrm{d}x \\ &= \tfrac{1}{4}\int \left(1 + 2\cos 2x + \tfrac{1}{2} + \tfrac{1}{2}\cos 4x\right)\mathrm{d}x \\ &= \tfrac{1}{4}\int \left(\tfrac{3}{2} + 2\cos 2x + \tfrac{1}{2}\cos 4x\right)\mathrm{d}x \end{aligned}$$

Now we can integrate the cosine terms:

$$\begin{aligned} &= \tfrac{1}{4}\left(\tfrac{3}{2}x + \frac{2\sin 2x}{2} + \frac{\tfrac{1}{2}\sin 4x}{4}\right) + C \\ &= \tfrac{3}{8}x + \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C \\ \therefore \quad & \int \cos^4 x\,\mathrm{d}x = \tfrac{1}{32}(12x + 8\sin 2x + \sin 4x) + C \end{aligned}$$

Odd powers of sine and cosine

Find $\int \sin^5 x \, dx$

The first thing to do in this case is to 'take out' a factor of $\sin x$

$$\int \sin^5 x \, dx = \int \sin x . \sin^4 x \, dx$$

and use the identity $\sin^2 x \equiv 1 - \cos^2 x$ to introduce the cosine function. This is necessary because it puts the integral into a form that we can integrate by substitution.

$$= \int \sin x (\sin^2 x)^2 \, dx$$

$$= \int \sin x (1 - \cos^2 x)^2 \, dx$$

$$= \int \sin x (1 - 2\cos^2 x + \cos^4 x) \, dx$$

$$= \int (\sin x - 2 \sin x \cos^2 x + \sin x \cos^4 x) \, dx$$

$$\int \sin^5 x \, dx = \int (\sin x - 2\sin x \cos^2 x + \sin x \cos^4 x) \, dx$$

The first term can be integrated by immediately. The other terms are to be integrated by substitution; put $u = \cos x$ in each case, so that

$$\int 2\sin x \cos^2 x \, dx = -\frac{2\cos^3 x}{3} + K_1$$

where K_1 is a constant, and

$$\int \sin x \cos^4 x \, dx = -\frac{\cos^5 x}{5} + K_2$$

where K_2 is another constant.

$$\therefore \int \sin^5 x \, dx = -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + C$$

Powers of tangent

Find $\int \tan^4 x \, dx$

Take a factor of $\tan^2 x$ from the integrand:

$$\int \tan^4 x \, dx = \int \tan^2 x \tan^2 x \, dx$$

Use the identity $\tan^2 x \equiv \sec^2 x - 1$:

$$= \int (\sec^2 x - 1) \tan^2 x \, dx$$

$$= \int \sec^2 x \tan^2 x \, dx - \tan^2 x \, dx$$

Replace the second term using the identity again:

$$= \int \sec^2 x \tan^2 x \, dx - (\sec^2 x - 1) \, dx$$

$$= \int \sec^2 x \tan^2 x \, dx - \sec^2 x + 1 \, dx$$

Use the substitution $u = \tan x$ to integrate the first term on the RHS.

$$\int \sec^2 x \tan^2 x \, dx = \frac{1}{3} \tan^3 x + K_1$$

The second term on the RHS can be integrated by inspection:

$$\int (-\sec^2 x + 1) \, dx = -\tan x + x + K_2$$

where K_1 and K_2 are constants.

Collecting these results we find:

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C$$

Summary

Integration can sometimes be simplified by substituting with a known trig identity. The following identities are usually useful:

Double angle for sine:

$$\sin 2x \equiv 2\sin x \cos x$$

To integrate powers of sine or cosine:

$$\sin^2 x \equiv \tfrac{1}{2}(1-\cos 2x) \qquad \text{and } \cos^2 x \equiv \tfrac{1}{2}(1+\cos 2x)$$

The integral of $\tan^2 x$ can be found from

$$\tan^2 x \equiv \sec^2 x - 1$$

Compound angles are useful for turning products into sums:

$$\begin{aligned} \sin(A \pm B) &\equiv \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &\equiv \cos A \cos B \mp \sin A \sin B \end{aligned}$$

This isn't a complete list...you may find other identities helpful too!

Test yourself...

Use trig identities to integrate the following. Write your answer in the simplest possible form.

1. $\int \cos^2 2x \, \mathrm{d}x$
2. $\int_0^{\frac{\pi}{12}} \tan^2 3x \, \mathrm{d}x$
3. $\int \sin 4x \cos 3x \, \mathrm{d}x.$ Hint: use $\sin(A+B)+\sin(A-B)$.
4. $\int_0^{\frac{\pi}{6}} 16 \sin x \cos x \, \mathrm{d}x$

Answers:

1. $\frac{1}{2}x + \frac{1}{8}\sin 4x + C$
2. $\frac{1}{3} - \frac{\pi}{12}$
3. $-\frac{1}{14}(\cos 7x + 7\cos x) + C.$
4. 2.