Integration with partial fractions

Calculus 9

INU0115/515 (Maths 2)

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Integrating $1/x$

Recall the result:

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

Note the modulus bars on the RHS — they're very important! It means negative limits can still be evaluated.

Evaluate $\displaystyle\int_{-3}^{-2} \frac{1}{x+1}\,\mathrm{d}x$

Use reverse chain rule:

$$\begin{aligned}
\int_{-3}^{-2} \frac{1}{x+1}\,\mathrm{d}x &= [\ln|x+1|]_{-3}^{-2} \\
&= \ln|-1| - \ln|-2| \\
&= \ln 1 - \ln 2 \\
&= -\ln 2
\end{aligned}$$

Partial fractions: recap

Partial fraction expansion

Express $\dfrac{x+3}{(x+1)(x+2)}$ as a partial fraction expansion.

The denominator contains linear factors so the expansion will have the form

$$\frac{x+3}{(x+1)(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+2}$$

Combine the terms on the RHS to get

$$\frac{x+3}{(x+1)(x+2)} \equiv \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

Equate the numerators:

$$x+3 \equiv A(x+2)+B(x+1)$$

Put $x=-1$ to get $A=2$.

Put $x=-2$ to get $B=-1$.Therefore

$$\frac{x+3}{(x+1)(x+2)} \equiv \frac{2}{x+1} - \frac{1}{x+2}$$

Integration using partial fractions

Partial fractions can sometimes be useful for simplifying an integral. Here's a simple example.

Using partial fractions

Find $\int \frac{x+3}{(x+1)(x+2)}\,\mathrm{d}x$

None of the methods (substitution, by parts) seen previously on the course are much help at the moment. However, using the previous example we can use partial fractions to express the integral as

$$\int \frac{x+3}{(x+1)(x+2)}\,\mathrm{d}x = \int \left(\frac{2}{x+1} - \frac{1}{x+2}\right)\mathrm{d}x$$

The terms on the RHS can now be integrated (reverse chain rule) to give:

$$\begin{aligned}\int \frac{x+3}{(x+1)(x+2)}\,\mathrm{d}x &= \int \frac{2}{x+1}\,\mathrm{d}x - \int \frac{1}{x+2}\,\mathrm{d}x \\ &= 2\ln|x+1| - \ln|x+2| + C\end{aligned}$$

Find $\int \frac{7x-16}{x(x-4)}\,\mathrm{d}x$

Split the integrand into partial fractions. The denominator contains linear factors, so:

$$\frac{7x-16}{x(x-4)} \equiv \frac{A}{x} + \frac{B}{x-4}$$

Therefore:

$$7x-16 \equiv A(x-4) + Bx$$

Choosing $x=0$ leads to $A=4$. And $x=4$ gives $B=3$.

The integral can be expressed instead as:

$$\int \frac{7x-16}{x(x-4)}\,\mathrm{d}x = \int \left(\frac{4}{x} + \frac{3}{x-4}\right)\mathrm{d}x$$

Integrate with reverse chain rule:

$$\int \frac{7x-16}{x(x-4)}\,\mathrm{d}x = 4\ln|x| + 3\ln|x-4| + C$$

Using logarithm rules to simplify expressions

Consider the answer obtained in the previous example:

$$\int \frac{x+3}{(x+1)(x+2)}\,\mathrm{d}x = 2\ln|x+1| - \ln|x+2| + C$$

The first two terms can be combined into a single expression using the rules of logarithms:

$$\begin{aligned}\int \frac{x+3}{(x+1)(x+2)}\,\mathrm{d}x &= \ln(x+1)^2 - \ln|x+2| + C \\ &= \ln\frac{(x+1)^2}{|x+2|} + C\end{aligned}$$

Now consider the constant C. It represents a value that we don't know. Let's write $C = \ln A$ where A is another constant. Therefore:

$$\begin{aligned} &= \ln\frac{(x+1)^2}{|x+2|} + \ln A \\ \therefore \int \frac{x+3}{(x+1)(x+2)}\,\mathrm{d}x &= \ln\frac{A(x+1)^2}{|x+2|}\end{aligned}$$

This answer is equivalent to that obtained in the previous example but in a more compact form.

Simplifying integrals with partial fractions

Find $\displaystyle\int \frac{5x^2-23x+6}{(2x+1)(x-2)^2}\,\mathrm{d}x$

Use partial fractions to simplify the integrand. It contains a repeated factor so the expansion is:

$$\begin{aligned}\frac{5x^2-23x+6}{(2x+1)(x-2)^2} &\equiv \frac{A}{2x+1}+\frac{B}{x-2}+\frac{C}{(x-2)^2}\\ &\equiv \frac{A(x-2)^2+B(2x+1)(x-2)+C(2x+1)}{(2x+1)(x-2)^2}\end{aligned}$$

The numerators are related by:

$$5x^2-23x+6 \equiv A(x-2)^2+B(2x+1)(x-2)+C(2x+1)$$

Put $x=2$ to get $C=-4$. Put $x=-\frac{1}{2}$ to get $A=3$. Try any other value of x and you'll find that $B=1$. This gives expressions that can be integrated with the reverse chain rule

$$\begin{aligned}\int \frac{5x^2-23x+6}{(2x+1)(x-2)^2}\,\mathrm{d}x &= \int \frac{3}{2x+1}+\frac{1}{x-2}-\frac{4}{(x-2)^2}\,\mathrm{d}x\\ &= 3\int (2x+1)^{-1}\,\mathrm{d}x+\int (x-2)^{-1}\,\mathrm{d}x-4\int (x-2)^{-2}\,\mathrm{d}x\\ &= \tfrac{3}{2}\ln|2x+1|+\ln|x-2|+\frac{4}{x-2}+K\end{aligned}$$

Definite integration and partial fractions

Evaluate $\int_3^4 \frac{1}{x^2-4}\,\mathrm{d}x$

We'll begin by expressing this using partial fractions.

$$\begin{aligned}
\int_3^4 \frac{1}{x^2-4}\,\mathrm{d}x &= \int_3^4 \frac{1}{(x-2)(x+2)}\,\mathrm{d}x \\
&= \int_3^4 \frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2}\,\mathrm{d}x \\
&= \left[\tfrac{1}{4}\ln|x-2| - \tfrac{1}{4}\ln|x+2|\right]_3^4 \\
&= \tfrac{1}{4}\left[\ln\left|\frac{x-2}{x+2}\right|\right]_3^4 \\
&= \tfrac{1}{4}\left(\ln\left|\frac{4-2}{4+2}\right| - \ln\left|\frac{3-2}{3+2}\right|\right) = \tfrac{1}{4}\left(\ln\left|\frac{2}{6}\right| - \ln\left|\frac{1}{5}\right|\right) \\
&= \tfrac{1}{4}(\ln\tfrac{1}{3} - \ln\tfrac{1}{5})
\end{aligned}$$

Using the rules of logarithms we can simplify this even further:

$$\int_3^4 \frac{1}{x^2-4}\,\mathrm{d}x = \tfrac{1}{4}\ln\left(\frac{1/3}{1/5}\right) = \tfrac{1}{4}\ln\tfrac{5}{3}$$

Test yourself...

Use partial fractions to integrate the following. Write your answer in the simplest possible form.

❶ $\int \frac{1}{x^2-9}\,\mathrm{d}x$

❷ $\int \frac{x}{(x-1)^2}\,\mathrm{d}x$

❸ $\int_1^2 \frac{8}{x(x^2+1)}\,\mathrm{d}x$

Answers:

❶ $\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C$ or $\frac{1}{6}\ln\left|\frac{A(x-3)}{x+3}\right|$

❷ $\ln|x-1| - \frac{1}{x-1} + C$

❸ $4\ln\frac{8}{5}$.