Integration by Substitution

Calculus 7

INU0115/515 (Maths 2)

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Recap: reverse chain rule

Integration using reverse chain rule

Find $\int (5x-1)^9 \, dx$

This requires us to integrate a power, so we add one and divide by the new power. We also divide by the derivative of the inner function $5x-1$:

$$\begin{aligned}\int (5x-1)^9 \, dx &= \frac{(5x-1)^{10}}{(10)(5)} + C \\ &= \frac{(5x-1)^{10}}{50} + C\end{aligned}$$

Easy! Reverse chain rule is quick method for doing integration. But other integrals need to be done differently. Let's tackle this integral using a different method.

Using a substitution to change the variable

Integration by changing the variable

Find $\int (5x-1)^9 \, dx$

Substitute $u = 5x - 1$. The integral becomes

$$\int u^9 \, dx$$

It already looks easier! But, we have to change the variable completely.

Differentiate u to get $\frac{du}{dx} = 5$.

Rearrange to get $dx = \frac{du}{5}$.

Now put this into the integral to finish the change of variable:

$$\int u^9 \frac{du}{5} = \frac{1}{5} \int u^9 \, du$$

(Notice how we took the $\frac{1}{5}$ outside the integral).

This can be integrated easily with respect to u.

$$\begin{aligned} \frac{1}{5} \int u^9 \, du &= \frac{1}{5} \times \frac{u^{10}}{10} + C \\ &= \frac{u^{10}}{50} + C \end{aligned}$$

Change back to the original variable using $u = 5x - 1$:

$$\therefore \int (5x-1)^9 \, dx = \frac{(5x-1)^{10}}{50} + C$$

Integration by substitution

Let's look at an example which cannot be solved using the reverse chain rule.

Integration by substitution

Find $\int 2x\left(x^2-1\right)^5 \, \mathrm{d}x$

Put $u = x^2 - 1$:

$$\int 2xu^5 \, \mathrm{d}x$$

We must change the variable completely. Differentiating the u w.r.t. x gives:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x \Rightarrow \mathrm{d}x = \frac{\mathrm{d}u}{2x}$$

Substituting this for the dx of the previous integral gives:

$$\int 2xu^5 \, \mathrm{d}x = \int 2xu^5 \frac{\mathrm{d}u}{2x} = \int u^5 \, \mathrm{d}u$$

Integrate with respect to u

$$= \frac{u^6}{6} + C$$

Change back to x:

$$\therefore \int 2x\left(x^2-1\right)^5 \, \mathrm{d}x = \frac{(x^2-1)^6}{6} + C$$

How does substitution work?

How can substitution make integration easier? In the previous example, the integrand contained x^2 term which was differentiated to get $2x$. This cancelled the $2x$ already in the integrand.

Integration by substitution relies on this kind of cancellation to simplify things.

Integration by substitution will work whenever we have an integral of the form

$$\int f\left[g(x)\right] g'(x)\,\mathrm{d}x$$

Making the substitution $u = g(x)$ (and so $\mathrm{d}u = g'(x)\,\mathrm{d}x$) leads to the integral

$$\int f(u)\,\mathrm{d}u$$

which is easier to compute.

Study the following example to see this cancellation again.

With trig functions

Find $\int \sin x \cos^3 x \, \mathrm{d}x$

Let $u = \cos x$.

Differentiate u to get $\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$ and so $\mathrm{d}x = -\frac{\mathrm{d}u}{\sin x}$.

Substituting these into the integral we get:

$$\begin{aligned} \int \sin x \cos^3 x \, \mathrm{d}x &= \int \sin x u^3 \left(-\frac{\mathrm{d}u}{\sin x}\right) \\ &= -\int u^3 \, \mathrm{d}u \end{aligned}$$

Now integrate with respect to u and change back to x at the finish.

$$\begin{aligned} &= -\frac{u^4}{4} + C \\ \therefore \int \sin x \cos^3 x \, \mathrm{d}x &= -\frac{\cos^4 x}{4} + C \end{aligned}$$

A harder substitution problem

Find $\int x(3x+2)^4\,\mathrm{d}x$

Using the substitution $u=3x+2$ we begin to change the integral to the new variable:

$$\int xu^4\,\mathrm{d}x$$

We can get $\mathrm{d}x$ by differentiating:

$$\frac{\mathrm{d}u}{\mathrm{d}x}=3 \qquad \Rightarrow \mathrm{d}x=\frac{\mathrm{d}u}{3}$$

The remaining x can be replaced by rearranging the equation $u=3x+2$ to get $x=\frac{1}{3}(u-2)$.

Making these substitutions gives the new integral:

$$\int \tfrac{1}{3}(u-2)u^4\times\frac{\mathrm{d}u}{3}=\tfrac{1}{9}\int(u-2)u^4\,\mathrm{d}u$$

This is integrated easily after expanding the brackets:

$$\begin{aligned}\tfrac{1}{9}\int(u-2)u^4\,\mathrm{d}u &= \tfrac{1}{9}\int(u^5-2u^4)\,\mathrm{d}u\\ &= \tfrac{1}{9}\left(\tfrac{1}{6}u^6-\tfrac{2}{5}u^5\right)+C\\ &= \tfrac{1}{54}u^6-\tfrac{2}{45}u^5+C\end{aligned}$$

Let's tidy this up by factorising:

$$=\frac{u^5}{270}(5u-12)+C$$

Change back to x:

$$\begin{aligned}&=\frac{(3x+2)^5}{270}(5(3x+2)-12)+C\\ &=\frac{(3x+2)^5(15x-2)}{270}+C\end{aligned}$$

Substitution and definite integrals

Substitution method with a definite integral

Evaluate $\int_0^{\frac{\pi}{6}} \cos x\sqrt{1-2\sin x}\,\mathrm{d}x$.

Use the substitution $u = 1-2\sin x$.
Differentiate and rearrange:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -2\cos x \quad \Rightarrow \mathrm{d}x = -\frac{\mathrm{d}u}{2\cos x}$$

Putting these into the integral gives:

$$= \int \cos x\sqrt{u} \times -\frac{\mathrm{d}u}{2\cos x}$$

$$= -\tfrac{1}{2}\int \sqrt{u}\,\mathrm{d}u$$

We must also change the limits of integration.

$x = \frac{\pi}{6} \Rightarrow u = 1-2\sin\frac{\pi}{6} = 0.$

$x = 0 \Rightarrow u = 1-2\sin 0 = 1.$

After changing the variable we have

$$\int_0^{\frac{\pi}{6}} \cos x\sqrt{1-2\sin x}\,\mathrm{d}x = -\tfrac{1}{2}\int_1^0 \sqrt{u}\,\mathrm{d}u$$

This can be integrated easily!

$$-\tfrac{1}{2}\int_1^0 u^{\frac{1}{2}}\,\mathrm{d}u = -\tfrac{1}{2}\left[\tfrac{2}{3}u^{\frac{3}{2}}\right]_1^0$$

$$= -\tfrac{1}{2}\left(0-\tfrac{2}{3}\right) = \frac{1}{3}$$

Note how we didn't have to change back to x to do evaluate the limits.

Trigonometric substitutions

Sometimes it is not obvious how to choose a substitution which will simplify an integral.

Consider the integral

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$

We cannot say 'let $u=1-x^2$' because there is no way for us to eliminate the dx part. (Try it!)

However if we make the substitution $x=\sin\theta$:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta}=\cos\theta \quad \Rightarrow \mathrm{d}x=\cos\theta\,\mathrm{d}\theta$$

and the integral becomes

$$\int \frac{1}{\sqrt{1-\sin^2\theta}}\times\cos\theta\,\mathrm{d}\theta=\int\frac{1}{\sqrt{\cos^2\theta}}\times\cos\theta\,\mathrm{d}\theta=\int\frac{\cos\theta}{\cos\theta}\,\mathrm{d}\theta=\int\mathrm{d}\theta$$

The integral is easy now.

$$\int \mathrm{d}\theta=\theta+C=\sin^{-1}x+C$$

How could we have known that $x = \sin\theta$ substitution would work in the previous example?

Unlike earlier substitions, it doesn't resemble any part of the integrand.

Recall from previous work with differentiation that we found the following result for inverse sine function:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

Perhaps we should have recognised or remembered this result! It shows that some algebraic expressions are connected to trig functions.

Since integration is the reverse of differentiation then we say:

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \sin^{-1} x + C$$

In this example, we might also have used the substitution $x = \cos\theta$; the resulting answer will have been just as valid.

Another trig substitution

Find $\displaystyle\int \frac{5}{25+x^2}\,\mathrm{d}x$

In this case we can make the substitution $x = 5\tan u$. Differentiate to get

$$\frac{\mathrm{d}x}{\mathrm{d}u} = 5\sec^2 u \quad \Rightarrow \mathrm{d}x = 5\sec^2 u\,\mathrm{d}u$$

Putting these into the integral gives:

$$\begin{aligned}\int \frac{5}{25+x^2}\,\mathrm{d}x &= \int \frac{5}{25+(5\tan u)^2} \times 5\sec^2 u\,\mathrm{d}u \\ &= \int \frac{25\sec^2 u\,\mathrm{d}u}{25+25\tan^2 u} \\ &= \int \frac{\sec^2 u}{1+\tan^2 u}\,\mathrm{d}u\end{aligned}$$

Recall the identity $1+\tan^2\theta = \sec^2\theta$ so that

$$\int \frac{5}{25+x^2}\,\mathrm{d}x = \int \frac{\sec^2 u}{\sec^2 u}\,\mathrm{d}u = \int \mathrm{d}u = u + C$$

But $u = \tan^{-1}(\frac{x}{5})$ so

$$\int \frac{5}{25+x^2}\,\mathrm{d}x = \tan^{-1}\left(\frac{x}{5}\right) + C$$

Test yourself...

Choose an appropriate substitution and find the following integrals.

❶ $\int 3x^2(x^3-4)^2\,\mathrm{d}x$

❷ $\int_0^1 \frac{4x}{8x^2+1}\,\mathrm{d}x$

❸ $\int 8x(1-x)^6\,\mathrm{d}x$

❹ $\int \frac{12}{1+x^2}\,\mathrm{d}x$

Answers:

❶ $\frac{1}{3}(x^3-4)^3+C$

❷ $\frac{1}{4}\ln 9$ (or $\frac{1}{2}\ln 3$).

❸ $-\frac{4}{27}(1-x)^7(1+7x)+C$

❹ $12\tan^{-1}x+C$