Reverse Chain Rule

Calculus 7

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Reversing the chain rule

In differentiation the chain rule is used to get the derivative of a composite function. For example

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{2x}) = 2\mathrm{e}^{2x}$$

Integration is the reverse of differentiation. Therefore, we could reverse this statement and write:

$$\int 2\mathrm{e}^{2x}\,\mathrm{d}x = \mathrm{e}^{2x}$$

(Ignore the constant for a moment). Dividing both sides by 2 we find

$$\int \mathrm{e}^{2x}\,\mathrm{d}x = \frac{\mathrm{e}^{2x}}{2} \tag{1}$$

Similarly, consider $\frac{\mathrm{d}}{\mathrm{d}x}[\sin(3x+2)] = 3\cos(3x+2)$. Reversing this process:

$$\int 3\cos(3x+2)\,\mathrm{d}x = \sin(3x+2)$$

Therefore:

$$\int \cos(3x+2)\,\mathrm{d}x = \frac{\sin(3x+2)}{3} \tag{2}$$

Can you a pattern for directly integrating the expressions given in (1) and (2)?

Reversing the chain rule

Here's another example of reversing the chain rule.

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x-5)^3 = (3)(2)(2x-5)^2$$

In words this process is 'Subtract 1 from the power. Multiply by the old power and multiply by the derivative of the inner function'.

Since integration is the reverse of differentiation:

$$\int (3)(2)(2x-5)^2 \, \mathrm{d}x = (2x-5)^3$$

(Again, let's ignore the constant C for a moment). Dividing both sides by the factor (3)(2) we get

$$\int (2x-5)^2 \, \mathrm{d}x = \frac{(2x-5)^3}{(3)(2)}$$

Notice that the (3) is the power and the (2) is the derivative of the inner function $(2x-5)$

$$\int (2x-5)^2 \, \mathrm{d}x = \frac{(2x-5)^3}{6}$$

The first term on the RHS is equivalent in words to 'Add 1 to the power. Divide by the new power and divide by the derivative of the inner function'. It is the reverse of applying the chain rule.

Reverse Chain Rule

The examples just seen illustrate a simple procedure for integrating composite functions. It is the reverse of how we'd carry out the chain rule in differentiation.

In words, the general procedure doing reverse chain rule is

'Integrate the outer function. Divide by the derivative of the inner function'

Here some examples. We'll include the constant from now on!

$$\int \sin 10x \, \mathrm{d}x = \frac{-\cos 10x}{10} + C$$

$$\int \mathrm{e}^{-4x} \, \mathrm{d}x = \frac{\mathrm{e}^{-4x}}{-4} + C$$

Reverse chain rule

Use the reverse chain rule to integrate $\int \frac{1}{(3-5x)^4}\,\mathrm{d}x$

First, write the integral in a more suitable form

$$\int (3-5x)^{-4}\,\mathrm{d}x$$

Now integrate. Use the rule for integrating powers (in this case, like x^{-4}); 'Add 1 to the power. Divide by the new power and divide by the derivative of the inner function'.

$$\int (3-5x)^{-4}\,\mathrm{d}x = \frac{(3-5x)^{-3}}{(-3)(-5)} + C = \frac{(3-5x)^{-3}}{15} + C$$

It can be written without the negative power like this:

$$\therefore \int \frac{1}{(3-5x)^4}\,\mathrm{d}x = \frac{1}{15(3-5x)^3} + C$$

When can reverse chain rule be used?

The reverse chain rule can only be used when the inner function is linear, which means it has the form $mx + c$ (where m and c are constants).

We cannot use the reverse chain rule when the inner function is NOT LINEAR!

For example, we *could not* use it to find $\int (1+x^2)^5 \, \mathrm{d}x$ because the inner function $1 + x^2$ is not linear.

Therefore

$$\int (1+x^2)^5 \, \mathrm{d}x \neq \frac{(1+x^2)^6}{6(2x)} + C$$

(You can check this by finding the derivative of the RHS - it will not be equal to the LHS!)

Test yourself

Which of the following can be integrated with the reverse chain rule?

1. $\int (1-3x)^4 \, dx$
2. $\int 2e^{-x} \, dx$
3. $\int \frac{1}{1-x^2} \, dx$
4. $\int \frac{1}{1-3x} \, dx$
5. $\int \sqrt{\sin x} \, dx$
6. $\int \cos(3x-\pi) \, dx$
7. $\int \sqrt{1+2x} \, dx$
8. $\int 2x \sin x \, dx$

Answers...

1. Yes
2. Yes
3. No
4. Yes
5. No
6. Yes
7. Yes
8. No

Further examples

Rational power

Find $\int \sqrt{2x+5}\,\mathrm{d}x$ with the reverse chain rule.

First rewrite without the square-root:

$$\begin{aligned}\int (2x+5)^{\frac{1}{2}}\,\mathrm{d}x &= \frac{(2x+5)^{\frac{3}{2}}}{(\frac{3}{2})(2)} + C \\ &= \frac{(2x+5)^{\frac{3}{2}}}{3} + C \\ &= \frac{\sqrt{(2x+5)^3}}{3} + C\end{aligned}$$

You can always check the answer by differentiating it! You should end up with the original integrand.

A definite integral

Evaluate $\int_0^1 (2x+1)^3 \, dx$

Using the reverse chain rule we see that

$$\begin{aligned} \int_0^1 (2x+1)^3 \, dx &= \left[\frac{(2x+1)^4}{(4)(2)} \right]_0^1 \\ &= \frac{1}{8} \left[(2x+1)^4 \right]_0^1 \\ &= \frac{1}{8} \left(3^4 - 1^4 \right) \\ &= \frac{1}{8} \times (81 - 1) \\ &= 10 \end{aligned}$$

Looking forward...

The reverse chain rule is a useful shortcut for simple situations. A related technique, called *integration by substitution*, will be demonstrated in the next lecture.

Integration of $f'(x)/f(x)$

Here is a result that we saw during our study of differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left|f(x)\right|\right]=\frac{f'(x)}{f(x)}$$

It's a consequence of differentiating logarithm functions using the chain rule.

Naturally, we can reverse this to produce an integration result:

$$\boxed{\int \frac{f'(x)}{f(x)}\,\mathrm{d}x=\ln\left|f(x)\right|+C} \tag{3}$$

This is a result we should learn to recognise when it appears!

Integration of $f'(x)/f(x)$

Find $\int \frac{2x}{x^2-9}\,\mathrm{d}x$.

This integral matches the form of equation (3) exactly.

Therefore:

$$\int \frac{2x}{x^2-9}\,\mathrm{d}x = \ln\left|x^2-9\right| + C$$

Integration of $f'(x)/f(x)$

Find $\int \frac{4\sin x}{1+\cos x}\,\mathrm{d}x$.

This integral almost fits the form of equation (3). We can rewrite it to make it the same:

$$\int \frac{4\sin x}{1+\cos x}\,\mathrm{d}x = -4\int \frac{-\sin x}{1+\cos x}\,\mathrm{d}x$$

We took out a factor of -4 so that the numerator is now the derivative of the denominator.

Therefore:

$$\int \frac{4\sin x}{1+\cos x}\,\mathrm{d}x = -4\ln|1+\cos x| + C$$

Test yourself

Use of reverse chain rule

Use the reverse chain rule to find these integrals:

1. $\int (1-3x)^4 \, dx$
2. $\int 2e^{-x} \, dx$
3. $\int \frac{1}{5+2x} \, dx$
4. $\int 2\cos(\frac{1}{3}x - \pi) \, dx$
5. $\int_0^{\frac{\pi}{9}} \sin 3x \, dx$
6. $\int_0^4 \frac{1}{\sqrt{1+2x}} \, dx$
7. $\int \frac{\cos x}{\sin x} \, dx$

Answers...

1. $-\frac{1}{15}(1-3x)^5 + C$
2. $-2e^{-x} + C$
3. $\frac{1}{2}\ln|5+2x| + C$
4. $6\sin(\frac{1}{3}x - \pi) + C$
5. $\frac{1}{6}$
6. 2
7. $\ln|\sin x| + C$