Definite Integration

Calculus 6

INU0115/515 (Maths 2)

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Definite integration

The integrals studied so far have generated functions containing a constant of integration. Those kinds of integrals are called *indefinite integrals*

$$\int f(x)\,\mathrm{d}x = F(x) + C$$

For example

$$\int (3x^2 + 10)\,\mathrm{d}x = x^3 + 10x + C$$

Now we introduce the *definite integral* which is written with numbers to the upper and lower right of the integral sign.

Definite integration

The expression

$$\int_a^b f(x)\,\mathrm{d}x$$

is called the definite integral of $f(x)$ from a to b. The numbers a and b are called the *lower limit* and *upper limit* respectively. They represent the values $x = a$ and $x = b$. Taken together, they are called the *limits* of the integral.

Evaluating a definite integral

Definite integrals are so-named because they have a definite numerical value. If we have:

$$\int f(x)\,\mathrm{d}x = A(x) + C$$

where C is the constant of integration, then the value of a definite integral is found by calculating the difference in the values of the RHS of the above equation for $x = a$ and $x = b$.

By definition it is given by:

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= [A(b)+C]-[A(a)+C] \\ &= A(b)-A(a)\end{aligned}$$

The constant of integration C always cancels from a definite integral, just leaving a number. So to evaluate a definite integral we just integrate $f(x)$ to obtain the function $A(x)$ and then find the difference between the values of $A(b)$ and $A(a)$.

The second fundamental theorem of calculus tells us that using the definite integral will calculate the area under a curve. We will study area problems soon, but for now, we'll get practice of evaluating definite integrals.

A simple definite integral

Evaluate $\int_1^3 3x^2\,\mathrm{d}x$

The indefinite integral for this is

$$\int 3x^2\,\mathrm{d}x = x^3 + C$$

That means we should write the definite integral like this:

$$\int_1^3 3x^2\,\mathrm{d}x = \left[x^3\right]_1^3$$

and then evaluate it at $x = 3$ and $x = 1$

$$\begin{aligned} \int_1^3 3x^2\,\mathrm{d}x &= 3^3 - 1^3 \\ &= 27 - 1 \\ \therefore \int_1^3 3x^2\,\mathrm{d}x &= 26 \end{aligned}$$

Another definite integral

Find the value of $\int_4^9 \frac{1}{\sqrt{x}}\,\mathrm{d}x$

Express the integrand as a power of x

$$\int_4^9 x^{-\frac{1}{2}}\,\mathrm{d}x$$

Use the 'power rule' to integrate:

$$\int_4^9 x^{-\frac{1}{2}}\,\mathrm{d}x = \left[2x^{\frac{1}{2}}\right]_4^9$$

Take the constant term outside the brackets:

$$\begin{aligned} \int_4^9 x^{-\frac{1}{2}}\,\mathrm{d}x &= 2\left[\sqrt{x}\right]_4^9 \\ &= 2(\sqrt{9}-\sqrt{4}) \\ &= 2(3-2) \\ \therefore \int_4^9 \frac{1}{\sqrt{x}}\,\mathrm{d}x &= 2 \end{aligned}$$

Yet another definite integral!

Find the value of $\int_{1}^{-2} (x^2 - 5)\,\mathrm{d}x$

Use the 'power rule' to integrate:

$$\begin{aligned}\int_{1}^{-2} (x^2 - 5)\,\mathrm{d}x &= \left[\tfrac{1}{3}x^3 - 5x\right]_1^{-2} \\ &= \tfrac{1}{3}\left[x^3 - 15x\right]_1^{-2}\end{aligned}$$

Substitute the limits (upper - lower).

$$\begin{aligned}\int_{1}^{-2} (x^2 - 5)\,\mathrm{d}x &= \tfrac{1}{3}((-8+30)-(1-15)) \\ &= \tfrac{1}{3}(22-(-14)) \\ &= \tfrac{1}{3}(36) \\ \therefore \int_{1}^{-2} (x^2 - 5)\,\mathrm{d}x &= 12\end{aligned}$$

Integrating trig functions and radians

Evaluate $\int_{\pi/6}^{\pi/3} 4\cos x\,\mathrm{d}x$

Remember that trigonometric functions are evaluated using radians.

$$\begin{aligned}
\int_{\pi/6}^{\pi/3} 4\cos x\,\mathrm{d}x &= [4\sin x]_{\pi/6}^{\pi/3} \\
&= 4(\sin\tfrac{\pi}{3} - \sin\tfrac{\pi}{6}) \\
&= 4\left(\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}\right) \\
&= 2\sqrt{3} - 2 \\
\therefore \int_{\pi/6}^{\pi/3} 4\cos x\,\mathrm{d}x &= 2(\sqrt{3} - 1)
\end{aligned}$$

Find the limit

Given that

$$\int_1^k (2x+5)\,\mathrm{d}x = 144$$

Find the possible values for k.

This is an integral equation which we can solve to find k.

Integrate the LHS:

$$\left[x^2+5x\right]_1^k = 144$$

Evaluate at the limits:

$$\begin{aligned}(k^2+5k)-(1^2+5(1)) &= 144\\ k^2+5k-6 &= 144\\ k^2+5k-150 &= 0\end{aligned}$$

Solve the quadratic to get $k=10$ and $k=-15$.

(You can check by evaluating $\int_1^{10}$ and $\int_1^{-15}$).

More terms makes it more complicated

Find the value of $\int_1^3 \left(x^4 + x^2 - \frac{2}{x} + \frac{1}{3x^2}\right) \mathrm{d}x$ to 4 significant figures.

First express the integrand in a form that can be integrated easily:

$$\int_1^3 \left(x^4 + x^2 - \frac{2}{x} + \frac{1}{3x^2}\right) \mathrm{d}x = \int_1^3 \left(x^4 + x^2 - 2x^{-1} + \tfrac{1}{3}x^{-2}\right) \mathrm{d}x$$
$$= \left[\tfrac{1}{5}x^5 + \tfrac{1}{3}x^3 - 2\ln x - \tfrac{1}{3}x^{-1}\right]_1^3$$

Now evaluate the difference between $x = 3$ and $x = 1$. Be careful and methodical here! It's useful to rewrite any negative powers.

$$= \left[\tfrac{1}{5}x^5 + \tfrac{1}{3}x^3 - 2\ln x - \tfrac{1}{3x}\right]_1^3$$
$$= \left(\tfrac{3^5}{5} + \tfrac{3^3}{3} - 2\ln 3 - \tfrac{1}{3(3)}\right) - \left(\tfrac{1^5}{5} + \tfrac{1^3}{3} - 2\ln 1 - \tfrac{1}{3(1)}\right)$$
$$= \left(\tfrac{243}{5} + 9 - 2\ln 3 - \tfrac{1}{9}\right) - \left(\tfrac{1}{5} + \tfrac{1}{3} - 0 - \tfrac{1}{3}\right)$$
$$= \tfrac{243}{5} + 9 - 2\ln 3 - \tfrac{1}{9} - \tfrac{1}{5}$$
$$= \tfrac{2578}{45} - 2\ln 3$$

Evaluating this on a calculator gives:

$$\int_1^3 \left(x^4 + x^2 - \frac{2}{x} + \frac{1}{3x^2}\right) \mathrm{d}x \approx 55.09 \text{ (to 4SF)}$$

Test yourself...

And now try to solve the following problems...

1. $\int_{1}^{2} 4x^3 \, dx$
2. $\int_{-1}^{1} x^3(1-x^2) \, dx$
3. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\sec^2 x \, dx$
4. Given that $\int_{0}^{k} (6x+2) \, dx = 120$, find the possible values of k.

Answers:

1. 15
2. 0
3. $2\sqrt{3}$
4. $k=6,\ k=-\frac{20}{3}$