Constant of Integration

Calculus 6

INU0115/515 (Maths 2)

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Objectives

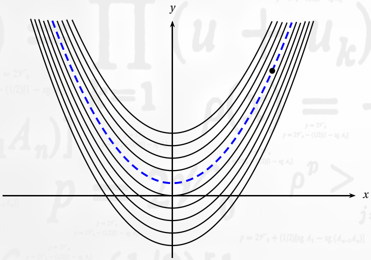
- By now you should be able to integrate simple functions; either using the rule for powers, or from a table of standard integrals.
- In all cases the integrated functions contain a constant of integration — usually denoted by C.
- In this presentation we'll see how to determine the value of the constant of integration.
- This is determined by extra information called a boundary condition (or an initial condition).

Boundary conditions

If we are given the gradient of a curve $\frac{dy}{dx}$ we can find the original curve y by integrating with respect to x.

For example, if $\frac{dy}{dx} = 2x$ we can integrate to find that $y = x^2 + C$.

The graph shows some of the family of curves described by $x^2 + C$.



How do we know which of this family was the original curve? We need some extra information - called a *boundary condition*.

If we knew the coordinates of a point on the original curve we can calculate C and identify the unique curve from the family.

For example, if the curve passes through the point $(3, 10)$ then we substitute $x = 3$ and $y = 10$ into the equation to get:

$$\begin{aligned} 10 &= 3^2 + C \quad \text{at the point (3,10)} \\ 10 - 9 &= C \\ C &= 1 \end{aligned}$$

The curve with gradient $2x$ which passes through the point (3,10) is therefore $y = x^2 + 1$

Applying a boundary condition

A curve has a gradient described by

$$\frac{dy}{dx} = 3x^2 - 2$$

Given that the curve passes through the point (-2,9) find the equation of the curve.

The curve is found by integration.

$$y = \int (3x^2 - 2)\,dx = x^3 - 2x + C$$

The boundary condition is $x = -2$, $y = 9$. Substituting into the equation gives:

$$\begin{aligned} 9 &= (-2)^3 - 2(-2) + C \\ 9 &= -8 + 4 + C \\ \therefore C &= 13 \end{aligned}$$

The curve satisfying the boundary condition is therefore $y = x^3 - 2x + 13$

Applying a boundary condition

The second derivative of a curve is given by

$$\frac{d^2y}{dx^2} = 4x^3$$

Given that $y = 1$ and $\frac{dy}{dx} = 2$ at $x = 1$, find the equation of the curve.

Since we are given the second derivative then we must integrate twice to obtain the curve equation.

$$\frac{dy}{dx} = \int 4x^3\,dx = x^4 + C$$

Use the first boundary condition is $x = 1$, $y' = 2$. Substituting into the equation gives:

$$2 = 1^4 + C \Rightarrow C = 1$$

So the first derivative is given by:

$$\frac{dy}{dx} = x^4 + 1$$

Integrating again gives:

$$\begin{aligned} y &= \int (x^4 + 1)\,dx \\ &= \tfrac{1}{5}x^5 + x + D \end{aligned}$$

where D is another constant. Apply the remaining boundary condition $x = 1$, $y = 1$:

$$1 = \tfrac{1}{5}(1^5) + 1 + D \Rightarrow D = -\tfrac{1}{5}$$

The curve satisfying the boundary condition is therefore $y = \frac{1}{5}x^5 + x - \frac{1}{5}$.

Initial conditions

If we're dealing with integration of a system where time t is a variable then the boundary condition may be specified at $t=0$. We usually refer to this information as an initial condition.

The acceleration of a particle is defined by

$$\frac{\mathrm{d}v}{\mathrm{d}t}=4t+3\mathrm{e}^{t}+1$$

Find $v(t)$ given that $v=0$ when $t=0$. Then calculate $v(2)$ to three decimal places.

Integrate with respect to t

$$\begin{aligned} v &= \int (4t+3\mathrm{e}^{t}+1)\,\mathrm{d}t \\ &= 2t^2+3\mathrm{e}^{t}+t+C \end{aligned}$$

Substitute $v=0$ and $t=0$ to show $C=-3$.

Therefore

$$v(t)=2t^2+3\mathrm{e}^{t}+t-3$$

Substitute $t=2$ to find $v(2)=29.167$ (to 3 DP).

Test yourself...

Let's practice finding those constants of integration!

1. Given $\dfrac{dy}{dx} = x^3$ and $(0, -2)$ is on the curve, find $y = f(x)$.
2. Given $\dfrac{dy}{dx} = \operatorname{cosec} x \cot x$ and $(\frac{5}{6}\pi, 4)$ is on the curve, find $y = f(x)$.
3. Given $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{x}}$ and $(\frac{1}{4}, \frac{1}{2})$ is on the curve, find $y = f(x)$.
4. Find $y = f(x)$ when $y'' = -\dfrac{1}{x^3}$ given that $x = 1$, $y = -2$ and $y' = 4$

Answers:

1. $y = \frac{1}{4}x^4 - 2$
2. $y = -\operatorname{cosec} x + 6$
3. $y = 4\sqrt{x} - \frac{3}{2}$.
4. $y = -\frac{1}{2x} + \frac{7}{2}x - 5$