Integration

Calculus 6

INU0115/515 (Maths 2)

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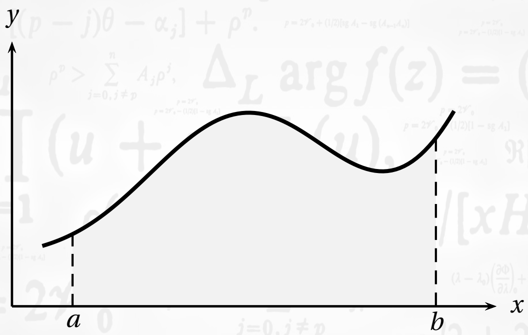
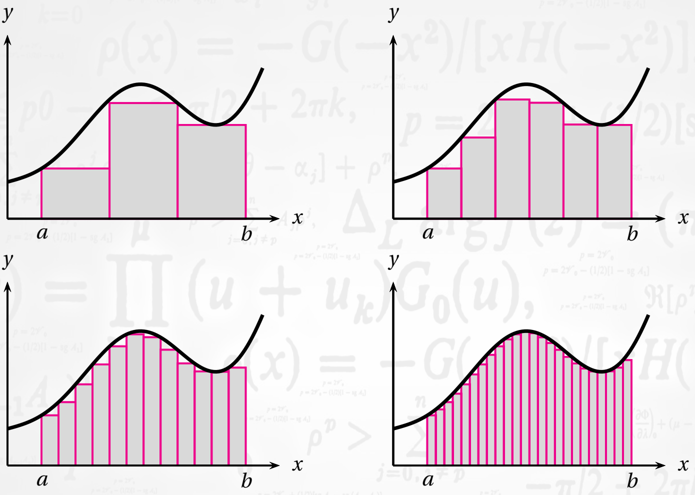
Area under a curve

Let's start with a function $y = f(x)$ and an interval bounded by the curve, the x-axis and the lines $x = a$ and $x = b$.



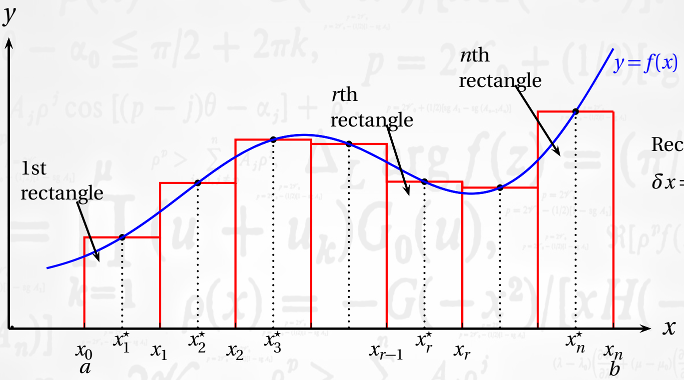
How can the shaded area be calculated? One way to do this is to divide up the region into lots of rectangular strips. Knowing the area of each rectangle will allow an estimate of the total area to be found.



Using more rectangles gives a better estimate of the actual shaded area.

It can be shown that the area under the curve is the limit of the sum of the rectangle areas.

Riemann Sums



Rectangle width

$$\delta x = \frac{b-a}{n}$$

The height of each rectangle is $f(x_r^\star)$.

The area of each rectangle is $f(x_r^\star)\delta x$.

The sum of areas of n rectangles is

$$S_n = f(x_1^\star)\delta x + f(x_2^\star)\delta x + \ldots + f(x_n^\star)\delta x$$

This sum, written using Sigma notation, is:

$$S_n = \sum_{r=1}^{n} f(x_r^\star)\delta x$$

The definite integral

The area under the curve is calculated by using more and more rectangles. In other words, we get a more accurate estimate as $n \to \infty$ (which also means $\delta x \to 0$).

$$\text{Area under curve} \quad = \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{r=1}^{n} f(x_r^\star)\delta x$$

It can be shown that this sum always converges to the same value as the area under the curve.

Since we are calculating the area as a limit between $x = a$ and $x = b$ as $\delta x \to 0$ then the short way of writing the sum is:

$$\boxed{\text{Area under curve, } A = \int_a^b f(x)\,\mathrm{d}x}$$

This is called a *definite integral* because it has a definite numerical value.

Using sums is not a very efficient way for calculating areas under curves!

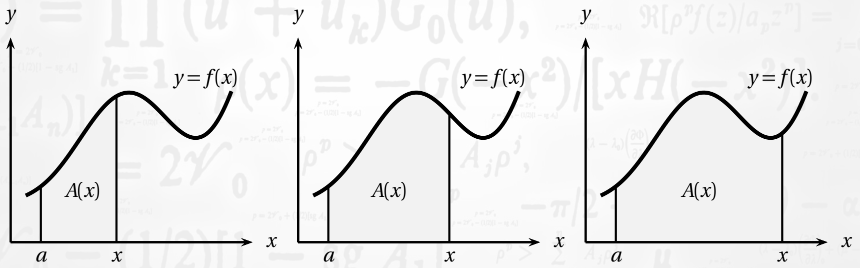
The area function

The area sum and the definite integral defines a function A which gives the area under a curve.

If we want the area between a fixed limit a and another limit x then we have to calculate

$$A(x) = \int_a^x f(x)\,\mathrm{d}x$$

Here's what changing the upper limit x does:

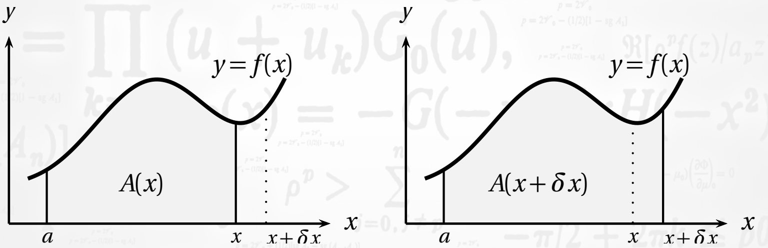


Think of that right edge as being like a curtain we can draw across the graph to increase or decrease the area function.

The area function

Next, we are going to investigate the relationship between the area function $A(x)$ and the curve $f(x)$. The goal is find a quicker method than calculating the sum of rectangles!

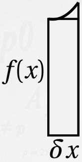
If we increase the value of x by a small amount δx then we get a small amount of area δA added to what we already have:



The small change in area is

$$\delta A = A(x + \delta x) - A(x) \tag{1}$$

The area function



Consider the small region δA again.

The area of δA is almost that of a rectangle. The area is approximately given as:

$$\delta A \approx f(x)\delta x \tag{2}$$

Using equations 1 and 2 means we can write:

$$f(x)\delta x \approx A(x+\delta x) - A(x)$$

Divide both sides by δx

$$f(x) \approx \frac{A(x+\delta x) - A(x)}{\delta x}$$

The approximation becomes exact if we let $\delta x \to 0$:

$$f(x) = \lim_{\delta x \to 0} \frac{A(x+\delta x) - A(x)}{\delta x}$$

The RHS is the 'first principles' definition of a derivative, so:

$$f(x) = \frac{\mathrm{d}A}{\mathrm{d}x}$$

This relationship tells us that the derivative of the area function is $f(x)$ - the function defining the curve.

In other words, we can find the area function $A(x)$ by reversing the process of differentiation starting with $f(x)$.

Fundamental theorems of calculus

First fundamental theorem of calculus

The area under a curve can be found by finding a function $A(x)$ whose derivative is $f(x)$. We say that $A(x)$ is an *antiderivative* of $f(x)$. Mathematicians also call $A(x)$ the *indefinite integral* of $f(x)$.

Consider the curve $y = x^2$. In this case we have $f(x) = x^2$. Here are some antiderivatives of x^2:

$$A(x) = \frac{x^3}{3}\ ,\ A(x) = \frac{x^3}{3} + 1\ ,\ A(x) = \frac{x^3}{3} - 10$$

We know they are antiderivatives because when we differentiate any of them we get $f(x)$. In fact any function of the form

$$A(x) = \frac{x^3}{3} + C$$

where C is a constant, is an antiderivative of x^2.

Fundamental theorems of calculus

If we know the antiderivative (the indefinite integral) of the curve $f(x)$ then we can use it to calculate the exact area under a curve. Like equation 1 and the picture above it, the area between two limits is found by evaluating the antiderivative $F(x)$ at the end points of the interval and finding the difference.

Second fundamental theorem of calculus

This tells us how to use the antiderivative to calculate the area under a curve. For a continuous function f on the interval $[a, b]$, where A is any antiderivative of f (with respect to x) then

$$\int_a^b f(x)\,\mathrm{d}x = A(b) - A(a) \tag{3}$$

This is a much more convenient system than using the Riemann sums that we started with! However, this method relies on being able to find an antiderivative for $f(x)$, which is not always possible (or easy).

Finding antiderivatives (indefinite integrals) will occupy us for now; we'll return to the problem of calculating areas when we've had practice finding indefinite integrals.

The reverse of differentiation

Differentiation is a process which obtains the derivative of a given function. For example, if we have the expression x^2 then:

$$x^2 \longmapsto \boxed{\text{differentiate}} \longmapsto 2x$$

If we are given the derivative and want to obtain the function it came from then this inverse process could be represented by:

$$2x \longmapsto \boxed{\text{anti}-\text{differentiate}} \longmapsto x^2$$

However, because any constant terms vanish when we differentiate, e.g.,

$$\left.\begin{array}{l} x^2-8 \\ x^2+2 \\ x^2-\pi \end{array}\right\} \longmapsto \boxed{\text{Differentiate}} \longmapsto 2x$$

then the inverse process must account for this.The inverse process is called *indefinite integration.*

$$2x \longmapsto \boxed{\text{Integrate}} \longmapsto x^2 + C \tag{4}$$

Indefinite integration always introduces a constant term C (called the constant of integration). To find the value of C we would need more information about the function.

Integration

The mathematical symbol[1] for integration is the integral sign $\int$.

So we would write the equation 4 as

$$\int 2x \, \mathrm{d}x = x^2 + C$$

The integral sign is always presented with dx. The expression between the integral sign and the dx is called the *integrand*.

You can think of the dx as indicating which variable we are integrating (the precise meaning is not important at this point).

Since integration is the reversal of the process of differentiation then we can find some of the rules of integration very easily.

[1] The integral sign is a is like a letter 's' which has been stretched vertically. The 's' indicates that integration is a way of calculating a sum!

Integrating powers

You should already know the following rules of differentiation. We will reverse them to find the equivalent rules for integration.

$$\frac{\mathrm{d}}{\mathrm{d}x}(ax) = a \quad \Rightarrow \boxed{\int a\,\mathrm{d}x = ax + C}$$

To integrate powers of x remember the rule for differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(ax^n) = anx^{n-1}$$

In words this means 'multiply by the power, and reduce the power of x by 1'. Note that the coefficient a does not affect the derivative - it remains as a coefficient. The same is true in integration; the coefficient does not affect the integration.

To integrate we reverse the rule and say 'add 1 to power of x and divide by the new power'.We write this as:

$$\boxed{\int ax^n\,\mathrm{d}x = \frac{ax^{n+1}}{n+1} + C \qquad \text{where } n \neq -1}$$

We cannot apply this rule when $n = -1$ because it would mean dividing by zero.

A really simple example

Find the integral of $2x^4$.

Apply the rule; add 1 to the power and divide by the new power.

$$\int 2x^4 \, \mathrm{d}x = \frac{2x^5}{5} + C$$

We can check that this is correct by differentiating; we should get the original integrand. Check:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x^5}{5} + C\right) = \frac{5(2)x^4}{5} + 0 = 2x^4$$

Integrating negative powers

Find the integral of $\dfrac{8}{x^2}$.

As with differentiation, we will first write this using a negative index

$$\int 8x^{-2} \, \mathrm{d}x = \frac{8x^{-1}}{-1} + C = -\frac{8}{x} + C$$

Integrating several terms (only one constant)

Integrate the expression $y^2 - 4y + 9$.

Notice here that the variable is y so we use dy. We integrate term by term.

$$\begin{aligned} \int (y^2 - 4y + 9)\,\mathrm{d}y &= \int y^2\,\mathrm{d}y - \int 4y\,\mathrm{d}y + \int 9\,\mathrm{d}y \\ &= \left(\frac{y^3}{3} + c_1\right) - \left(\frac{4y^2}{2} + c_2\right) + (9y + c_3) \\ &= \frac{y^3}{3} - 2y^2 + 9y + c_1 - c_2 + c_3 \end{aligned}$$

But since $c_1 - c_2 + c_3$ is equal to another constant, say, C we can simply write:

$$\int (y^2 - 4y + 9)\,\mathrm{d}y = \frac{y^3}{3} - 2y^2 + 9y + C$$

This example shows that when we integrate an expression, *we only need to use one constant of integration.* This is usually done at the end of the process

Test yourself...

Find the following integrals.

1. $\int x^7 \, \mathrm{d}x$
2. $\int \sqrt{x} \, \mathrm{d}x$
3. $\int \left(\frac{2}{x^2} - 6x + 1 \right) \mathrm{d}x$

Answers:

1. $\int x^7 \, \mathrm{d}x = \frac{1}{8} x^8 + C$
2. $\int \sqrt{x} \, \mathrm{d}x = \int x^{\frac{1}{2}} \, \mathrm{d}x = \frac{x^{\frac{3}{2}}}{(3/2)} = \frac{2}{3} x^{\frac{3}{2}} + C$
3. $\int \left(\frac{2}{x^2} - 6x + 1 \right) \mathrm{d}x = \int \left(2x^{-2} - 6x + 1 \right) \mathrm{d}x = -\frac{2}{x} - 3x^2 + x + C$

The constant C must always be included, otherwise the answer is wrong!

Some more integrals

To integrate x^{-1} we recall that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \dfrac{1}{x}$. Therefore:

$$\boxed{\int \frac{1}{x}\,\mathrm{d}x = \ln x + C}$$

Inspection of a table of derivatives (see your notes on Differentiation) also leads to the following results:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x \qquad \Rightarrow \boxed{\int \cos x\,\mathrm{d}x = \sin x + C}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x \qquad \Rightarrow \boxed{\int \sin x\,\mathrm{d}x = -\cos x + C}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x \qquad \Rightarrow \boxed{\int \sec^2 x\,\mathrm{d}x = \tan x + C}$$

We can obtain many other results by inspecting a table of derivatives.

List of standard integrals

$$\int a\,\mathrm{d}x = ax + C$$

$$\int ax^n\,\mathrm{d}x = \frac{ax^{n+1}}{n+1} + C,\ n \neq -1$$

$$\int \frac{1}{x}\,\mathrm{d}x = \ln x + C$$

$$\int \sin x\,\mathrm{d}x = -\cos x + C$$

$$\int \cos x\,\mathrm{d}x = \sin x + C$$

$$\int \sec^2 x\,\mathrm{d}x = \tan x + C$$

$$\int \operatorname{cosec} x \cot x\,\mathrm{d}x = -\operatorname{cosec} x + C$$

$$\int \sec x \tan x\,\mathrm{d}x = \sec x + C$$

$$\int \operatorname{cosec}^2 x\,\mathrm{d}x = -\cot x + C$$

$$\int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x + C$$

$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C$$

$$\int \sinh x\,\mathrm{d}x = \cosh x + C$$

$$\int \cosh x\,\mathrm{d}x = \sinh x + C$$

Using the standard integrals

Integrate $\frac{3}{x}+\sqrt{x}-4\sec x\tan x$.

We can rewrite the question using the integral sign:

$$\int\left(3x^{-1}+x^{\frac{1}{2}}-4\sec x\tan x\right)\mathrm{d}x$$

Integrating term by term:

$$\begin{aligned}\int\left(3x^{-1}+x^{\frac{1}{2}}-4\sec x\tan x\right)\mathrm{d}x &= 3\ln x+\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-4\sec x+C\\ &= 3\ln x+\frac{2x^{\frac{3}{2}}}{3}-4\sec x+C\\ &= 3\ln x+\frac{2\sqrt{x^3}}{3}-4\sec x+C\end{aligned}$$

where C is an arbitrary constant.

Test yourself...

Use your knowledge of standard integrals to answer the following questions.

1. $\displaystyle\int 10x\,\mathrm{d}x$
2. $\displaystyle\int \frac{6}{x^3}\,\mathrm{d}x$
3. $\displaystyle\int 10^x\,\mathrm{d}x$
4. $\displaystyle\int 2\sec^2 x\,\mathrm{d}x$

Answers:

1. $5x^2 + C$
2. $-\dfrac{3}{x^2} + C$
3. $\dfrac{1}{\ln 10}10^x.$
4. $2\tan x + C$