Small increments

Calculus 13

INU0115/515 (Maths 2)

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Objectives

In this presentation we're going to look at one of the applications of partial differentiation.

- Absolute changes to a function.
- Relative changes to a function (percentage errors)
- How do these influence our predictions or measurements? (sensitivity analysis)

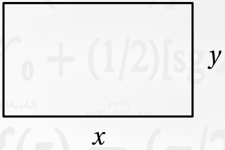
By this stage you should be able to comfortably find all first partial derivatives for a given function.

Small changes

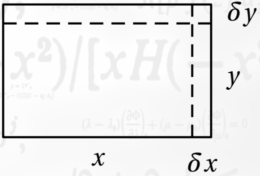
Consider a rectangle with a length and width of x and y.

The area A of the rectangle is

$$A = xy$$



If we increase the dimensions by small amounts δx and δy then we'll also increase the area by a small amount δA.



The area of this new rectangle is related to the original one by

$$A + \delta A = (x + \delta x)(y + \delta y)$$

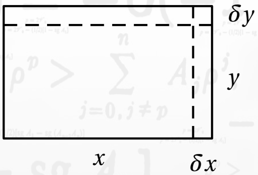
We're going to investigate the relationship between δA and the changes δx and δy.

$$A + \delta A = (x + \delta x)(y + \delta y)$$

Multiply the brackets out on the RHS:

$$A + \delta A = xy + y\delta x + x\delta y + \delta x\delta y$$

Each of the terms on the RHS corresponds to one of the smaller regions in the picture below.



Now since $A = xy$ then those terms cancel out:

$$\delta A = y\delta x + x\delta y + \delta x\delta y$$

If the changes δx and δy are small then the product $\delta x\delta y$ is the smallest term on the RHS.

To a good approximation we can say:

$$\delta A \approx y\delta x + x\delta y \tag{1}$$

Take a look at the formula for area we started with and notice that the two partial derivatives are

$$\frac{\partial A}{\partial x} = y \quad \text{and} \quad \frac{\partial A}{\partial y} = x$$

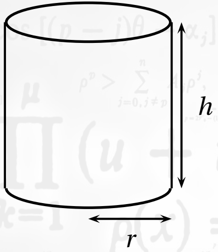
Therefore we express equation (1) in these terms:

$$\delta A \approx \frac{\partial A}{\partial x}\delta x + \frac{\partial A}{\partial y}\delta y \tag{2}$$

This gives the change in area as a relationship between the partial derivatives and the small changes themselves.

Small changes (again)

Let's choose another shape and try again.



Consider the case of a cylinder with radius r and height h.

The volume V is given by

$$V = \pi r^2 h$$

We can write down partial derivatives of the volume:

$$\frac{\partial V}{\partial h} = \pi r^2 \quad \text{and} \quad \frac{\partial V}{\partial r} = 2\pi r h$$

If we make small changes to the radius (δr) and height (δh), then the volume will also change by a small amount δV

$$V + \delta V = \pi(r + \delta r)^2(h + \delta h)$$

Now expand the brackets:

$$\begin{aligned} V + \delta V &= \pi(r^2 + 2r\delta r + (\delta r)^2)(h + \delta h) \\ &= (\pi r^2 + 2\pi r\delta r + \pi(\delta r)^2)(h + \delta h) \\ V + \delta V &= \pi r^2 h + 2\pi rh\delta r + \pi h(\delta r)^2 + \pi r^2 \delta h + 2\pi r\delta r\delta h + \pi(\delta r)^2 \delta h \end{aligned}$$

We know $V = \pi r^2 h$ so it will cancel from both sides:

$$\delta V = 2\pi rh\delta r + \pi h(\delta r)^2 + \pi r^2 \delta h + 2\pi r\delta r\delta h + \pi(\delta r)^2 \delta h$$

The δ quanties are small, which means terms like $(\delta r)^2$ or $\delta r\delta h$ are even smaller. We're going to ignore them now but doing so means the relationship is no longer exact:

$$\delta V \approx 2\pi rh\delta r + \pi r^2 \delta h$$

The volume of a cylinder is $V = \pi r^2 h$.

The partial derivatives are:

$$\frac{\partial V}{\partial r} = 2\pi rh \text{ and } \frac{\partial V}{\partial h} = \pi r^2$$

Let's examine that formula for δV for the cylinder:

$$\delta V \approx 2\pi rh\delta r + \pi r^2 \delta h$$

It contains the partial derivatives of V:

$$\delta V \approx \frac{\partial V}{\partial r}\delta r + \frac{\partial V}{\partial h}\delta h$$

This means that the small change in volume is related to the small changes in the radius and height and we can can calculate the overall change using the partial derivatives; we don't have to expand the brackets in the way we did here!

Chain rule for small changes

It doesn't matter whether we start with a rectangle, cylinder or some other formula.

For a function of two variables $u = u(x, y)$ small changes in x and y will produce a change in the overall value of u given by

$$\delta u \approx \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y$$

For a function of three variables $u = u(x, y, z)$ we add another 'link' to the chain:

$$\delta u \approx \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

For functions of four or more variables the pattern continues in the same way.

Small changes

Consider a cylinder of radius 3 cm and height 2 cm. How does the volume change when the radius is increased by 0.1 cm and the height is decreased by 0.2 cm?

We saw that the change in volume is given by

$$\delta V \approx 2\pi rh\delta r + \pi r^2 \delta h$$

We substitute the radius and height $r = 3$, $h = 2$ into this along with the small changes $\delta r = 0.1$ and $\delta h = -0.2$.

$$\begin{aligned} \delta V &\approx 2\pi(3)(2)(0.1) + \pi(3^2)(-0.2) \\ &\approx 1.2\pi - 1.8\pi \\ &\approx -0.6\pi \end{aligned}$$

The volume *decreases* by approximately 0.6π (about $1.88\,\mathrm{cm}^3$).

A right angle triangle

A right angled triangle has shorter sides of length $a = 3$ cm and $b = 4$ cm. Use the chain rule to calculate the change to the hypotenuse if a is increased by 0.1 cm and b is decreased by 0.2 cm.

The hypotenuse length is given by $c = (a^2 + b^2)^{\frac{1}{2}}$. The partial derivatives with respect to the two variables are:

$$\frac{\partial c}{\partial a} = \frac{a}{\sqrt{a^2 + b^2}}, \quad \frac{\partial c}{\partial b} = \frac{b}{\sqrt{a^2 + b^2}}$$

It is given that $\delta a = 0.1$ and $\delta b = -0.2$. The chain rule for this problem is:

$$\delta c \approx \frac{\partial c}{\partial a}\delta a + \frac{\partial c}{\partial b}\delta b$$

Substitution gives:

$$\begin{aligned}
\delta c &\approx \frac{a}{\sqrt{a^2 + b^2}}\delta a + \frac{b}{\sqrt{a^2 + b^2}}\delta b \\
&\approx \frac{3}{\sqrt{3^2 + 4^2}}(0.1) + \frac{4}{\sqrt{3^2 + 4^2}}(-0.2) \\
&\approx (0.6)(0.1) + (0.8)(-0.2) \\
\delta c &\approx -0.1
\end{aligned}$$

The small changes give an approximate 0.1 cm *decrease* in the hypotenuse.

Percentage errors

A force F between two masses M and m separated by a distance r is given by

$$F = \frac{GMm}{r^2}$$

where G is a constant. Suppose errors of $\pm 1\%$ are possible in measurements of M, m and r, find the maximum possible error in the calculated value of F.

The chain rule for this situation is

$$\delta F \approx \frac{\partial F}{\partial M}\delta M + \frac{\partial F}{\partial m}\delta m + \frac{\partial F}{\partial r}\delta r$$

The measurement errors are $\delta M = \pm 0.01M$, $\delta m = \pm 0.01m$ and $\delta r = \pm 0.01r$.

Substituting these, with the partial derivatives, into the formula gives:

$$\delta F \approx \frac{Gm}{r^2}(\pm 0.01M) + \frac{GM}{r^2}(\pm 0.01m) + \frac{-2GMm}{r^3}(\pm 0.01r)$$

The variables can be simplified in each term:

$$\delta F \approx \frac{GMm}{r^2}(\pm 0.01) + \frac{GMm}{r^2}(\pm 0.01) + \frac{-2GMm}{r^2}(\pm 0.01)$$

In each term we have the original expression for F:

$$\begin{aligned}\delta F &\approx F(\pm 0.01) + F(\pm 0.01) - 2F(\pm 0.01) \\ &\approx F(\pm 0.01 \pm 0.01 \mp 0.02)\end{aligned}$$

To get *maximum possible error* combine the ±terms to give the biggest values:

$$\delta F \approx \pm 0.04 F$$

Therefore a maximum error of ±4% in the calculated value of F is possible.

Sensitivity Analysis

The chain rule for small changes is useful for quantifying the contributions of each variable change to the overall change in the function.

Sensitivity analysis

Suppose a cylinder with radius $r = 10$ cm and $h = 1$ cm is to be constructed by a machine. Is the volume more sensitive to construction errors in the radius or height?

The volume of the cylinder is $V = \pi r^2 h$ so that the chain rule gives:

$$\delta V \approx 2\pi r h \delta r + \pi r^2 \delta h$$

Substituting the radius and height gives $\delta V \approx 20\pi \delta r + 100\pi \delta h$

This formula tells us that the biggest contribution to volume (for small errors of the same size) comes from the second term; the volume is more sensitive to errors in the height δh.

If the design is changed to $r = 1$ cm, $h = 10$ cm then we find $\delta V \approx 20\pi \delta r + \pi \delta h$.

The volume is now much more sensitive to errors in the radius δr.

Test yourself...

Let's finish with some further practice of small increment anatysis.

1. Consider the function
$$P = \frac{4a^2 b^5}{c^2}$$
Write down an expression for δP in terms of a, b and c.
2. In (1) find the approximate percentage change in P when a and b both decrease by 2% and c decreases by 5%.
3. The current I in a circuit with voltage V and total resistance R is
$$I = \frac{V}{R}$$
When $V = 12$ and $R = 100$, is the current more sensitive to changes in voltage or resistance?

Answers:

1. $\delta P \approx \frac{8ab^5}{c^2}\delta a + \frac{20a^2 b^4}{c^2} - \frac{8a^2 b^5}{c^3}\delta c$
2. P decreases by $\approx 4\%$.
3. $\delta I \approx \frac{1}{100}\delta V - \frac{12}{100^2}\delta R$. The current is much more sensitive to changes in voltage with these values.