SECOND DERIVATIVES

CALCULUS 13

INU0115/515 (MATHS 2)

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Objectives

In the previous presentation you learned how to find partial derivatives of functions with two or more variables.

Things can get complicated after that!

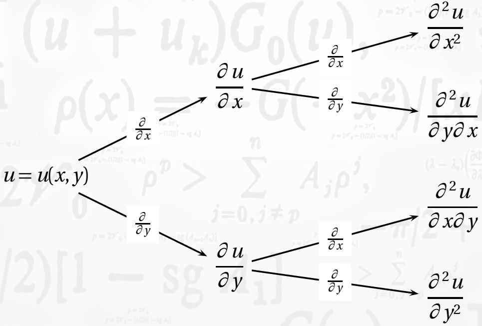
For each partial derivative there are at least two further derivatives that can be calculated. In this presentation we'll learn how to do that and how derivative notation works.

Finally, we'll see a useful result called *mixed derivative theorem* which links all derivatives of the same order together.

Second derivatives

We can differentiate each of the first derivatives to obtain second derivatives. Since the functions contains two or more variables then we have a greater number of second derivatives.

For example, if we begin with $u = u(x, y)$ (so u is a function of x and y), the following picture shows all possible first and second derivatives.



Second derivatives

Find all the second partial derivatives of

$$u = x^2 \sin y - e^x + y^2$$

The first derivatives are

$$\frac{\partial u}{\partial x} = 2x \sin y - e^x \text{ and } \frac{\partial u}{\partial y} = x^2 \cos y + 2y$$

Each of these functions has two derivatives. Differentiating the first partial derivative gives:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} = 2 \sin y - e^x$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = 2x \cos y$$

Differentiating the other first derivative gives:

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} = -x^2 \sin y + 2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y} = 2x \cos y$$

Mixed derivative theorem

In the previous example we found two of the second derivatives were the same. This was not a coincidence! The order of differentiation (x then y, or y then x) has no bearing on the end result[1].

Mixed derivative theorem

The mixed derivative theorem states that for a function $u(x, y)$ and its partial derivatives then

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

provided the function and its derivatives are continuous.

The mixed derivative tells us, for example, about the equality of these third derivatives

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial^3 u}{\partial y \partial x^2}$$

[1] This is true provided that the function and its derivatives are continuous everywhere.

Using mixed derivative theorem

Given the function

$$f(x,y) = \frac{x^3 \sin 4x}{\cos 2x} + 3xy^2$$

Calculate the second derivative $\dfrac{\partial^2 f}{\partial y \partial x}$.

To find the required derivative we should differentiate with x and then y.

The first term will make this unnecessarily complicated.

It's better to differentiate with y first and then x:

$$\frac{\partial f}{\partial y} = 6xy \quad \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = 6y$$

And by the mixed derivative theorem we can say

$$\therefore \frac{\partial^2 f}{\partial y \partial x} = 6y$$

Notation

The partial derivative notation can be messy when it comes to denoting higher order derivatives for functions of many variables. So far, we've been using Leibniz notation for our derivatives.

There is a shorter system – subscript notation – for denoting derivatives.

Given the function $f(x, y)$ the first derivatives can be represented by:

$$\frac{\partial f}{\partial x} \text{ or } f_x$$

$$\frac{\partial f}{\partial y} \text{ or } f_y$$

Things are a little different when denoting higher order derivatives...

Leibniz vs Subscript notation

Consider the derivative $\dfrac{\partial f}{\partial y}$.

Now, if we differentiate this with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

We read the order of differentiation from right-to-left in the second derivative (y then x).

In subscript notation: starting with f_y and differentiating with respect to x:

$$(f_y)_x = f_{yx}$$

This is the same derivative as before but the order is read left-to-right in the subscripts. This is a confusing difference but both systems are widely used so we should get used it :-)

The Heat Equation

Joseph Fourier (1768 — 1830)

The Heat Equation in one dimension:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where u is the temperature at time t at distance x along a one dimensional object (the c^2 is a constant).

The French mathematician Joseph Fourier found a solution to this PDE using a technique which transformed it into a ordinary differential equation similar to the simple harmonic motion (vibration) equation that we'll see later on the course.

Fourier expressed his solution in the form of an infinite series of sines and cosines. That method, called a Fourier Series, is a useful technique with wider applications.

The Wave Equation

James Clerk Maxwell (1831 — 1879)

The Wave Equation in one dimension:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where u is the displacement at time t at distance x from an origin (the c^2 is a constant). Many physical phenonomena such as light, sound (and waves) can be modelled using the wave equation.

Scottish physicist James Clerk Maxwell showed that magnetism, light and electricity were manifestations of the same underlying phenomena: electromagnetism and electromagnetic waves. Maxwell's equations are a series of PDEs whose solutions explained a huge number of previously unrelated observations and experiments.

Test yourself...

Let's try some problems based on partial second derivatives.

1. $u = \dfrac{10x^3}{y}$ Find *all* of the second derivatives.
2. Given $f(x, y, z)$. To calculate $\dfrac{\partial^3 f}{\partial x \partial z \partial y}$ — in what order must we differentiate ?
3. Given
$$u = y^2 \ln(3y^4 - 1) + 12x^3y^2$$
Use mixed derivative theorem to find u_{yx}.

Answers:

1. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{60x}{y}, \quad \dfrac{\partial^2 u}{\partial y^2} = \dfrac{20x^3}{y^3}, \quad \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x} = -\dfrac{30x^2}{y^2}$
2. y then z then x (read right-to-left)
3. Find u_{xy} instead (differentiate with x first). $u_{xy} = 72x^2y = u_{yx}$