Partial Differentiation

Calculus 13

INU0115/515 (Maths 2)

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Functions of many variables

The functions and derivatives which have been analysed on this course have been functions of one variable, e.g. $y = f(x)$.

Many of the equations describing interesting phenomena in the real world - sound waves, electromagnetism, vibrations, heat transfer and so on - are dependent on more than one variable.

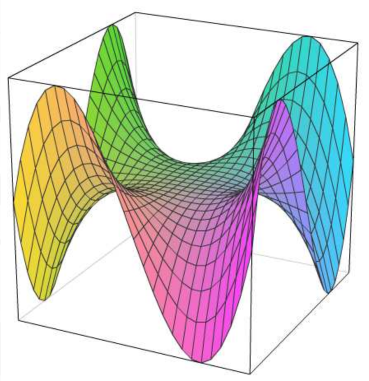
For example, the temperature in warm room with a door open to the cold weather outside, depends on the temperature inside and outside the room, the location within the room and the time elapsed since the door was opened.

Analysis of functions with many variables requires the rules of calculus to be extended. Studying how these functions change with respect to each variable requires partial differentiation to be carried out.

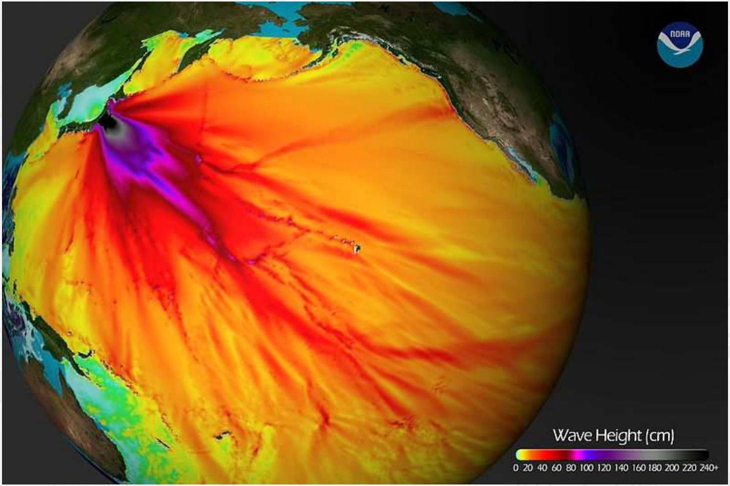
Functions of many variables

The function $y = f(x)$ defines a curve in 2D space. A function of two variables, $z = f(x, y)$, defines a surface in 3D space.

For example, the function $z = xy^3 - x^3y$ looks like this:



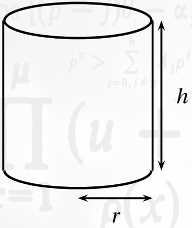
The function $u = f(x, y, t)$ defines how a surface changes with time.



Numerical simulation of wave heights for the Pacific Ocean tsunami (March 11th 2011). The wave height at a given location and time is a function of many variables including earthquake magnitude, water depth, distance from the source and intervening geography.

Partial derivatives

Consider a cylinder of radius r and height h.



The volume V is given by

$$V = \pi r^2 h$$

Consider how V changes as h changes (and we treat r as a constant value):

$$\left[\frac{\mathrm{d}V}{\mathrm{d}h}\right]_{r\,\text{constant}} = \pi r^2$$

We can also find how V changes with respect to r, when h is held constant.

$$\left[\frac{\mathrm{d}V}{\mathrm{d}r}\right]_{h\,\text{constant}} = 2\pi r h$$

These two derivatives are called partial derivatives.

Mathematicians use *curly-d* notation to indicate partial differentiation.

$$\frac{\partial V}{\partial h} = \pi r^2 \quad \text{and} \quad \frac{\partial V}{\partial r} = 2\pi r h$$

Finding partial derivatives

Given the function

$$f(x,y) = 8x^2 + 10y^3 - 5xy^2 + 10$$

Find the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Use the 'normal' rules for differentiation on each variable whilst treating the other as a constant.

First, differentiate x whilst treating y as a constant:

$$\frac{\partial f}{\partial x} = 16x - 5y^2$$

Now, differentiate y whilst treating x as a constant:

$$\frac{\partial f}{\partial y} = 30y^2 - 10xy$$

Finding partial derivatives

Given the function

$$u = x^2 \sin y + x e^y$$

Find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$.

Differentiate with respect to x, treating y (and therefore $\sin y$ and e^y as a constants).

$$\frac{\partial u}{\partial x} = 2x \sin y + e^y$$

Differentiate with respect to y, treating x as a constant.

$$\frac{\partial u}{\partial y} = x^2 \cos y + x e^y$$

Product rule

Using the product rule

Given the function

$$u = xy\sin x$$

Find all of the first partial derivatives.

Factorise the function:

$$u = y(x\sin x)$$

Differentiate with respect to x. Since y is constant it doesn't affect the differentiation of x.

$$\begin{aligned}\frac{\partial u}{\partial x} &= y \times (x\cos x + (1)\sin x) \\ &= y(x\cos x + \sin x)\end{aligned}$$

The derivative with respect to y is simpler:

$$\frac{\partial u}{\partial y} = (x\sin x)(1) = x\sin x$$

Chain rule

Using the chain rule

Given the function

$$u = (3x + y^2)^4$$

Find all of the first partial derivatives.

Recall the chain rule:

$$(\text{Outer function})' \times (\text{Inner function})'$$

Differentiate with respect to x, treating y (and functions of it) as constants.

$$\frac{\partial u}{\partial x} = 4(3x + y^2)^3 \times 3 = 12(3x + y^2)^3$$

Now differentiate with respect to y, treating x as a constant.

$$\frac{\partial u}{\partial y} = 4(3x + y^2)^4 \times 2y = 8y(3x + y^2)^3$$

Quotient rule

Using the quotient rule

Given the function

$$u = \frac{2x - y}{x + y}$$

Find all of the first partial derivatives.

We apply the quotient rule in the usual way, remembering to treat variables not being differentiated as constants.

The partial derivative with respect to x

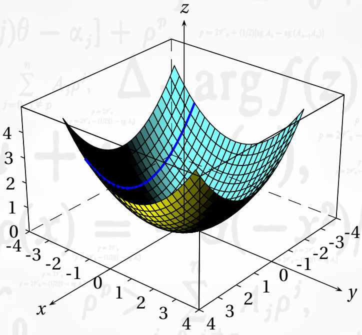
$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{(x+y)(2) - (2x-y)(1)}{(x+y)^2} \\ &= \frac{2x + 2y - 2x + y}{(x+y)^2} = \frac{3y}{(x+y)^2} \end{aligned}$$

The partial derivative with respect to y

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{(x+y)(-1) - (2x-y)(1)}{(x+y)^2} \\ &= \frac{-x - y - 2x + y}{(x+y)^2} = -\frac{3x}{(x+y)^2} \end{aligned}$$

What is a partial derivative?

Consider the function $z = x^2 + y^2$. This is a surface in 3D space:



If we slice the surface along the line $y = -2$ then we see that its boundary is a curve $z = x^2 + 4$.

The gradient of this curve is represented by the partial derivative $\dfrac{\partial z}{\partial x} = 2x$.

First principles

Recall when we first met differentiation the rules we learned (powers, standard derivatives, etc) were based on an underlying definition called 'first principles':

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Derivatives for functions of many variables can also be derived using an equivalent definition.

Consider the general function of two variables $z = f(x, y)$. This time we can calculate the derivative with respect to x or y. This is achieved by considering a small change in x whilst holding y constant:

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

This is the first partial derivative with respect to x. We could also calculate the corresponding partial derivative with respect to y (so holding x constant):

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$$

The quick method we've been using — *Use the 'normal' rules for differentiation on each variable whilst treating the other as a constant* — is a consequence of the definition.

Let's end this section with a quick demonstration of the first principles method.

First principles

From first principles derive the partial first derivatives of $z = 3x^2 + y^2$.

The partial derivative with respect to x:

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \lim_{\delta x \to 0} \frac{3(x+\delta x)^2 + y^2 - (3x^2 + y^2)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{3(x^2 + 2x\delta x + \delta x^2) + y^2 - (3x^2 + y^2)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{3x^2 + 6x\delta x + 3\delta x^2 + y^2 - 3x^2 - y^2}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{6x\delta x + 3\delta x^2}{\delta x} \\
&= \lim_{\delta x \to 0} 6x + 3\delta x \\
\frac{\partial z}{\partial x} &= 6x
\end{aligned}$$

Note the convention of writing $(\delta x)^2$ as δx^2; this is so we don't have a confusing number of brackets!

First principles

From first principles derive the partial first derivatives of $z = 3x^2 + y^2$.

Now vary y while keeping x constant.

From first principles:

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \lim_{\delta y \to 0} \frac{3x^2 + (y+\delta y)^2 - (3x^2 + y^2)}{\delta y} \\
&= \lim_{\delta y \to 0} \frac{3x^2 + y^2 + 2y\delta y + \delta y^2 - 3x^2 - y^2}{\delta y} \\
&= \lim_{\delta y \to 0} \frac{2y\delta y + \delta y^2}{\delta y} \\
&= \lim_{\delta y \to 0} 2y + \delta y \\
\frac{\partial z}{\partial y} &= 2y
\end{aligned}$$

Test yourself...

You should know enough about partial derivatives to answer these. Find all the first partial derivatives for the following functions.

1. $u = 8x^3y^2 + 1$
2. $u = (2x+3y)^4$
3. $u = x^3\tan(xy^2)$
4. $u = \dfrac{10x^3z^4}{y^3}$

Answers:

1. $\dfrac{\partial u}{\partial x} = 24x^2y^2$ and $\dfrac{\partial u}{\partial y} = 16x^3y$
2. $\dfrac{\partial u}{\partial x} = 8(2x+3y)^3$ and $\dfrac{\partial u}{\partial y} = 12(2x+3y)^3$
3. $\dfrac{\partial u}{\partial x} = 3x^2\tan(xy^2) + x^3y^2\sec^2(xy^2)$ and $\dfrac{\partial u}{\partial y} = 2x^4y\sec^2(xy^2)$.
4. $\dfrac{\partial u}{\partial x} = \dfrac{30x^2z^4}{y^3}$, $\dfrac{\partial u}{\partial y} = -\dfrac{30x^3z^4}{y^4}$ and $\dfrac{\partial u}{\partial z} = \dfrac{40x^3z^3}{y^3}$.