Direction fields

Calculus 12

INU0115/515 (Maths 2)

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Objectives

From earlier work you should now be familiar with the following:

- You know what a differential equation is.
- You can solve simple first order equations by separation of variables.

You should also take a moment to think about the following question:

- What does the quantity $\frac{\mathrm{d}y}{\mathrm{d}x}$ represent?
- How does this quantity relate to the line equation $y = mx + c$?

In this presentation we're going to explore the geometry of first order differential equations.

For the purposes of the exam — this topic will not be assessed. The material does provide useful background for more advanced studies of differential equations.

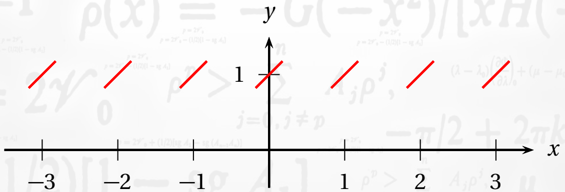
Sketching solutions of differential equations

Let's see what the solution to a differential equation looks like on a graph. Consider the equation given by:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y$$

This equation shows us that the *gradient* at any point on the graph is equal to the y value at that point. For example at every point where $y = 1$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$.

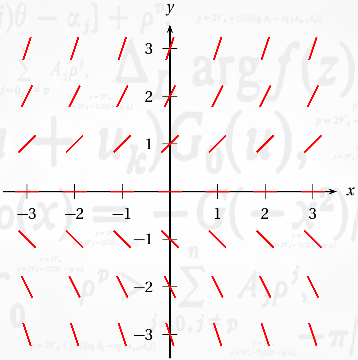
We can draw small line segments, with a slope of 1, to represent the gradient of the solution at these points.



We can cover the entire graph with lines like these!

Direction fields

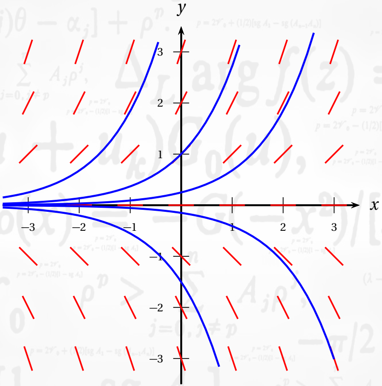
Given the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y$ we can sketch the *direction field*:



This type of graph is also called a *vector field* or a *slope field*; it is a picture of the general solution for the differential equation.

Particular solutions on the direction field

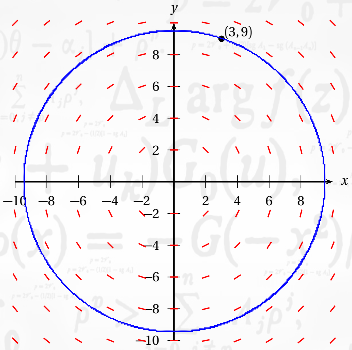
Given a direction field of a differential equation we can use it to sketch particular solutions. Just follow the direction of the line segments!



Do you recognise the shape of the blue curves? They are exponential curves of the form $y = Ae^x$.

A more complicated direction field

Sketch the direction field for the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.



The particular solutions to this equation are circles; this can be shown by separating the variables and solving.

The circle shows the solution associated with the given boundary condition $y(3) = 9$.

Graphing complicated direction fields

Only the simplest direction fields can be plotted by pencil and paper.

For more complicated differential equations it will be necessary to use a dedicated plotter or other software.

- `https://goo.gl/Hr8Kwz` (Mine!)
- `https://goo.gl/0w8bw1`

...or search for others online.

What's the point?

Direction fields give a picture of the solutions to a differential equation *without doing integration.*

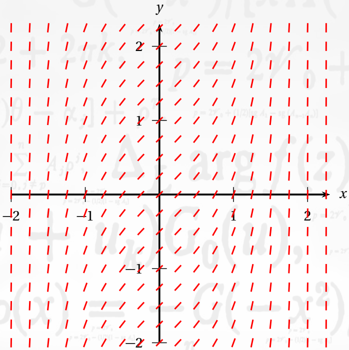
This is very useful in cases where we have to solve a differential equation but the integration is difficult or impossible.

For example the equation

$$\frac{dy}{dx} = e^{x^2}$$

doesn't have solutions we can obtain by integration; but we can see what the solutions are by examining the direction field!

Here's the direction field for $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{x^2}$.



How can the computer draw the solution curves?

If we know the gradient at a particular point we can draw a straight line segment to find another point. There is an algorithm to do this known as Euler's method; it is a type of numerical integration (like Trapezium or Simpson's rule) but designed to solve differential equations.

See `https://en.wikipedia.org/wiki/Euler_method` for an introduction.