GROWTH AND DECAY

CALCULUS 12

INU0115/515 (MATHS 2)

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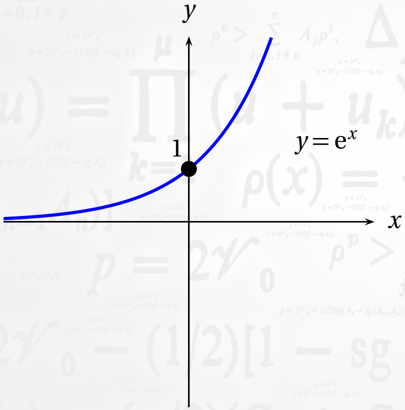
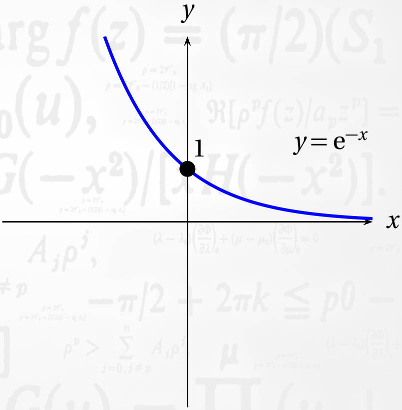
Introduction

Some of the simplest systems that can be modelled by differential equations are those involving growth and decay. In this presentation we'll revise some prior mathematics before studying differential equations in this specific context.

- Exponential growth and decay
- Graphs of exponential functions
- Constant of proportionality
- Forming and solving differential equations
- Case studies:
  - Population growth
  - Radioactive decay
- Shifted growth and decay

Exponential graphs





Exponential growth

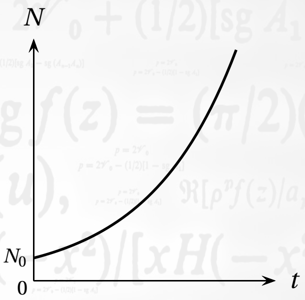
Solutions to differential equations of growth usually have the independant variable as time t and the base invariably turns out to be base e.

The solutions often have the form

$$N = N_0 e^{kt}$$

where N is the dependant variable and k is a constant which controls how quickly the growth happens in time t.

The graph is still exponential growth like the previous one!



At $t = 0$ we have $N = N_0$.

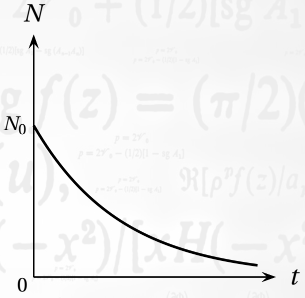
Therefore N_0 represents the initial value of N.

Exponential decay

Solutions to differential equations of decay turn out to have the form

$$N = N_0 e^{-kt}$$

where N is the dependant variable and k is a constant which controls how quickly the decay happens in time t.



At $t = 0$ we have $N = N_0$.

The graph is still exponential decay like the previous one.

Therefore N_0 represents the initial value of N.

Constant of proportionality

If a is proportional to b the relationship is written

$$a \propto b$$

This means that a increases as b increases. Furthermore, it means there is a constant k such that

$$a = kb$$

This relationship is linear; a graph of a against b would show a straight line of gradient k.

If a is inversely proportional to b the relationship is written

$$a \propto \frac{1}{b}$$

This means that a increases as b decreases (or vice-versa). In this case it means there is a constant k such that

$$a = \frac{k}{b}$$

The constant k is called a constant of proportionality.

Forming a differential equation

Given a description of a problem you should be able to form a differential equation to describe it.

Write down differential equations describing the following situations.

1. At time t, the rate of increase of a population with P people is proportional to the number present at that time.
2. The density ρ of the atmosphere decreases with height h. The rate of change is proportional to height.
3. The rate of increase of ice thickness x on a lake at time t is inversely proportional to the thickness of ice already present.
4. The gradient of a curve is inversely proportional to the product of its coordinates.
5. Newton's law of cooling states that an object at temperature T will change at a rate proportional to the difference between object temperature and the temperature of the surroundings T_0.

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1. $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$
2. $\frac{\mathrm{d}\rho}{\mathrm{d}h} = -kh$
3. $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{k}{x}$
4. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{xy}$
5. $\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_0)$

Exponential growth

A scientist initially counts 160 bacteria in a sample of water. Assuming the number of bacteria N increases at a rate proportional to the number present, write down a differential equation connecting N and the time t. If the rate of increase of the number is initially 80 per hour, how many are there after 3 hours?

The differential equation for this

$$\frac{\mathrm{d}N}{\mathrm{d}t} \propto N$$

Rewriting with a constant of proportionality:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = kN$$

We are also given *initial conditions*

$N = 160$ and $\frac{\mathrm{d}N}{\mathrm{d}t} = 80$ when $t = 0$.

Use them to find k:

$$80 = 160k \Rightarrow k = \tfrac{1}{2}$$

The differential equation becomes:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \tfrac{1}{2}N$$

Solve by separating the variables:

$$\int \frac{1}{N}\mathrm{d}N = \int \tfrac{1}{2}\,\mathrm{d}t$$

And integrate:

$$\ln N = \tfrac{1}{2}t + C$$

Exponentiate both sides:

$$N = e^{\frac{1}{2}t + C} = Ae^{\frac{1}{2}t}$$

The constant A can be found using $N = 160$ when $t = 0$.

$$160 = Ae^{\frac{1}{2}\times 0} \;\Rightarrow 160 = A$$

The particular solution is therefore $N = 160e^{\frac{1}{2}t}$.

This solution can be used to find the number of bacteria at any given time t.

When $t = 3$ hours:

$$N = 160e^{\frac{1}{2}\times 3} \approx 717.07$$

So there are about 717 bacteria present after 3 hours.

Case Study: Population growth

In a simple model of population growth the number of members P increases at a rate proportional to the number present at time t.

$$\frac{\mathrm{d}P}{\mathrm{d}t} \propto P \qquad \therefore \frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

If there are initially P_0 members then:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\ ,\ P(0) = P_0$$

Separate the variables:

$$\int \frac{1}{P}\,\mathrm{d}P = \int k\,\mathrm{d}t$$

Integrate:

$$\ln P = kt + C$$

Substitute $P = P_0$ and $t = 0$:

$$\ln P_0 = C$$

Put this back into the equation:

$$\ln P = kt + \ln P_0$$

Next, we'll combine the logarithm terms together:

$$\begin{aligned} \ln P - \ln P_0 &= kt \\ \ln\left(\frac{P}{P_0}\right) &= kt \end{aligned}$$

Remove the logarithm by exponentiating both sides:

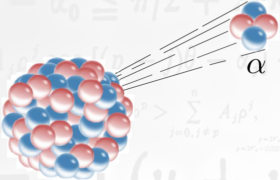
$$\frac{P}{P_0} = \mathrm{e}^{kt}$$

The population function $P(t)$ is therefore given by

$$\boxed{P = P_0\mathrm{e}^{kt}}$$

To find the value of k we would need a further piece of information; another measurement of P at another time, or the value of P' at some time.

Case Study: Radioactive decay



This is a depiction of alpha decay - one of several types of radioactivity.

Radioactive decay is a process in which an unstable atom loses energy by emitting particles.

In doing so the original atom is transformed into an atom of another element. For example, a certain kind of Uranium is transformed into Lead by this process.

It is impossible to predict when a particular atom will decay. However, if large numbers of atoms are present then the *rate* at which decay occurs is predictable.

The physical law which governs radioactivity states that if there are N radioactive atoms present in a sample at time t then the rate of decay is directly proportional to N.

Formulating the decay equation

Given that there are N radioactive atoms present in a sample at time t then the rate of decay is directly proportional to N.

The rate of change is negative because N is decreasing as t increases. Therefore the differential equation governing radioactivity is:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -kN$$

(where k is a positive constant of proportionality whose value depends on the type of atom).

Suppose we know that there are N_0 atoms present at $t = 0$. This is an initial condition (a type of boundary condition) which will provide information about the constant of integration.

If we want to make predictions about the value of N at some future (or past) time t then we must solve the differential equation.

Therefore the initial value problem to be solved is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -kN \; , \; N(0) = N_0$$

Solving the decay equation

Solution is by separating the variables and integrating:

$$\begin{aligned} \int \frac{1}{N}\,\mathrm{d}N &= -\int k\,\mathrm{d}t \\ \ln N &= -kt + C \end{aligned}$$

Using the rules of logs and indices we can express this more compactly:

$$N = \mathrm{e}^{-kt+C} = A\mathrm{e}^{-kt}$$

Where we have changed the form of the constant of integration ($A = \mathrm{e}^C$).

Apply the initial condition $N = N_0$ at $t = 0$:

$$N_0 = A\mathrm{e}^0 \Rightarrow A = N_0$$

The solution to the differential equation is given by:

$$N = N_0\mathrm{e}^{-kt}$$

Half-life of a radioactive substance

It is useful to know the time taken for half of the atoms present to decay. We can find this very easily from the solution.

$$N = N_0 e^{-kt}$$

We are trying to find the time t for which $N = \frac{1}{2} N_0$:

$$\begin{aligned} \tfrac{1}{2} N_0 &= N_0 e^{-kt} \\ \tfrac{1}{2} &= e^{-kt} \end{aligned}$$

Take natural logs of both sides and simplify:

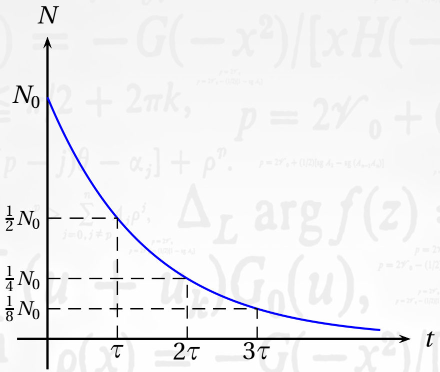
$$\ln \tfrac{1}{2} = -kt \quad \therefore t = -\frac{\ln \frac{1}{2}}{k}$$

This special value of t, called the *half-life*, is represented by the Greek letter τ ('tau') and it can be simplified further:

$$\tau = (\ln 2)/k$$

Uses of the Half-life

Potassium-40 has a half-life of 1.25 billion years and it has been used to date the age of the Earth (about 4.5 billion years old). Iodine-128 has a half life of 25 minutes and is used to measure the performance of the thyroid gland in the human body.



The number of atoms in a substance (and therefore the total mass) undergoing radioactive decay will fall to half of its initial value after a period of τ has passed.

After 2τ it will have halved again so that it is one-quarter of the initial value.

After 3τ has passed the number of atoms will be just one-eighth of the initial value.

Shifted growth and decay

This type of situation occurs when the growth (or decay) function is modified to include some other growth or decay factor.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky + c$$

This equation describes many different situations. Some examples include:

- A growing population including a constant immigration or emigration.
- Newton's law of heating/cooling.
- Compound interest on a bank account with deposits or withdrawals.
- Mixing problems (rate of change depends on input and output rates).

Population growth with emigration

Assume a city population is growing naturally (exponentially) but 1,000 people per year are leaving the city. Given that the natural growth constant is $k = 0.01$ and that there are currently 200,000 people in the city, calculate the population in 10 years and 20 years from the present.

The differential equation for this is:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.01P - 1000$$

Separate the variables and integrate:

$$\begin{aligned} \int \frac{1}{0.01P - 1000}\,\mathrm{d}P &= \int \mathrm{d}t \\ 100\ln|0.01P - 1000| &= t + C \end{aligned}$$

Now we must find the value of the constant C...

$$100\ln|0.01P - 1000| = t + C$$

Given $t = 0$ (the present) and $P = 200\,000$.

Substitute the values to get:

$$C = 100\ln 1000$$

The particular solution is.

$$100\ln|0.01P - 1000| = t + 100\ln 1000$$

We are trying to get a function of the form $P = f(t)$.

Start rearranging the equation to find it...

Bring the log terms to the LHS

$$\begin{aligned} 100\ln|0.01P-1000|-100\ln 1000 &= t \\ 100(\ln|0.01P-1000|-\ln 1000) &= t \end{aligned}$$

Use log rules to simplify:

$$\begin{aligned} 100\ln\left(\frac{0.01P-1000}{1000}\right) &= t \\ \ln\left(\frac{0.01P-1000}{1000}\right) &= \frac{t}{100} \end{aligned}$$

Make P the subject:

$$\begin{aligned} \frac{0.01P-1000}{1000} &= \mathrm{e}^{t/100} \\ 0.01P-1000 &= 1000\mathrm{e}^{t/100} \\ 0.01P &= 1000\mathrm{e}^{t/100} \end{aligned}$$

The final function for $P(t)$ is

$$P = 100\,000 + 100\,000\mathrm{e}^{t/100}$$

The population of the city is described by

$$P = 100\,000 + 100\,000 e^{t/100}$$

A quick check: $t = 0$ gives $P = 100\,000 \times (1 + 1) = 200\,000$ — as expected!

After $t = 10$ years we will find $P = 210\,517$.

After $t = 20$ years we calculate that $P = 222\,140$.

Test yourself...

Try to answer the following questions...

1. Given $P = 2000e^{2t}$ where t is time; is this a growth or decay equation?
2. Solve $\dfrac{dQ}{dt} = -kQ$ given that $Q = Q_0$ at $t = 0$.
3. In (2) given that $Q = 3$ at $t = 1$ and $Q = 0.25$ at $t = 4$, find the values of Q_0 and k.
4. The rate of decrease of gravitational force F with distance r from a mass is inversely proportional the cube of the distance. Write down a differential equation for this situation.
5. Solve $\dfrac{dx}{dt} = 0.1x + 2$ given that $x(0) = 50$. What happens to x as $t \to \infty$?

Answers:

1. Growth; because of the positive exponent.
2. $Q = Q_0 e^{-kt}$
3. $k = \frac{1}{3}\ln 12$; $Q_0 = 3 \times 12^{\frac{1}{3}}$.
4. $\dfrac{dF}{dr} = -\dfrac{k}{r^3}$
5. $x = 70e^{t/10} - 20$. As $t \to \infty$, $x \to \infty$.