# DIFFERENTIAL EQUATIONS

## CALCULUS 12

INU0115/515 (MATHS 2)

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# Introduction

## Differential equation

Equations which contain one or more derivatives of the dependent variable are called differential equations.

Differential equations occur in many scientific models describing phenomena in science and engineering. They occur where the rate of change of a function is related to the function itself.

A simple of example of a differential equation, which we shall study in more detail later, is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y$$

To solve a differential equation means to find a function satisfying the differential equation. Can you think of function $f(x)$ which satisfies this differential equation?

# Famous examples

Lots of diverse phenomena in across many branches of mathematics, science and engineering are modelled by differential equations. Some famous examples:

Radioactive decay The rate at which a mass $m$ of radioactive material is decaying depends on the amount of material present at that time $t$:

$$\frac{\mathrm{d}m}{\mathrm{d}t} \propto m$$

Newton's law of cooling The temperature $T$ of an object changes at a rate proportional to the difference in temperatures between the object and surroundings (at $T_0$):

$$\frac{\mathrm{d}T}{\mathrm{d}t} \propto (T_0 - T)$$

Newton's second law Force $F$ is equal to the product of mass $m$ and acceleration. The acceleration is a second derivative of displacement $x$:

$$F = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

# Types of differential equation

On this course we will study *ordinary differential equations* which have solutions that are functions of a single variable (e.g. $y = f(x)$).

## Order of a differential equation

The *order* of the differential equation is the number of the highest derivative.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 7y \qquad t^3\frac{\mathrm{d}^4 V}{\mathrm{d}t^4} - \left(\frac{\mathrm{d}V}{\mathrm{d}t}\right)^2 = \sin t$$

First order Second order Fourth order

## Linear and nonlinear differential equation

If any of the derivatives is raised to a power other than one, then the differential equation is said to be *nonlinear*.

In the list above, the last differential equation is nonlinear. In general, nonlinear differential equations are much more difficult to solve than linear equations.

On this course we will restrict our attention to solving certain types of first-order, linear, ordinary differential equations.

# Solving differential equations

Here is a simple differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

Solving a differential equation means to find a function which satisfies the equation. In this case, we can integrate to find the solution

$$y = x^2 + C$$

You can check this is a solution by differentiating!

Here's another differential equation:

$$\sec x \frac{\mathrm{d}y}{\mathrm{d}x} = 4$$

Rewrite the left hand side using the definition of $\sec x$:

$$\frac{1}{\cos x}\frac{\mathrm{d}y}{\mathrm{d}x} = 4$$

and then cross multiply:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4\cos x$$

Now integrate to get to get the solution.

$$y = 4\sin x + C$$

# Solution by separation of variables

Not all differential equations are so easily integrated.

We will now consider first order linear differential equations which are a product or quotient of separate functions of $x$ and $y$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y) \qquad \text{or} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x)}{g(y)}$$

In either case we can separate the variables so that:

$$\frac{1}{g(y)}\,\mathrm{d}y = f(x)\,\mathrm{d}x \qquad \text{and} \qquad g(y)\,\mathrm{d}y = f(x)\,\mathrm{d}x$$

To solve we must integrate both sides:

$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x \qquad \text{and} \qquad \int g(y)\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

By integrating each side of these equations we'll get a function which a solution to the original differential equation. Then, if necessary, we can apply boundary conditions to find the value of constant of integration.

## Solving a differential equation by separating the variables

Find the general solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -3x^2y^2$$

Cross multiply by $\mathrm{d}x$ and divide by $-y^2$ to separate the variables.

$$-\frac{1}{y^2}\,\mathrm{d}y = 3x^2\,\mathrm{d}x$$

Now we can do the integrations with $x$ and $y$ on different sides of the equation:

$$-\int \frac{1}{y^2}\,\mathrm{d}y = \int 3x^2\,\mathrm{d}x$$

Integrate each side with respect to the variable:

$$\frac{1}{y} = x^3 + C$$

Rearrange to get:

$$y = \frac{1}{x^3 + C}$$

This function is the *general solution* of the differential equation.

# Verifying (checking!) a solution

How can we check that

$$y = \frac{1}{x^3 + C}$$

is indeed the solution to $\frac{\mathrm{d}y}{\mathrm{d}x} = -3x^2y^2$ ?

By differentiating the function $y$ — we'll find get expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and it should match the differential equation at the start.

So

$$y = (x^3 + C)^{-1}$$

Using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -(x^3 + C)^{-2} \times 3x^2 = -3x^2(x^3 + C)^{-2}$$

We can rewrite this in terms of $y$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -3x^2\left[(x^3 + C)^{-1}\right]^2 = -3x^2y^2$$

And so $y = \frac{1}{x^3 + C}$ is definitely a solution.

## Solving a differential equation by separating the variables

Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{3x^2}{y}$$

Separate the variables so that we can integrate with $x$ and $y$ on different sides:

$$y\,dy = 3x^2\,dx$$

Now add the integral signs:

$$\int y\,dy = \int 3x^2\,dx$$

Integrate each side with respect to the variable:

$$\tfrac{1}{2}y^2 = x^3 + C$$

This function is the *general solution* to the differential equation. This can be verified by carrying out implicit differentiation! A neater form for the above solution is:

$$y^2 = 2x^3 + D$$

where we labelled $2C$ as another constant called $D$.

Also: we don't always need to make $y$ the subject of the general solution.

## Solving a differential equation by separating the variables

Find the general solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y$$

Separate the variables so that we can integrate with $x$ and $y$ on different sides:

$$\frac{1}{y}\,\mathrm{d}y = \mathrm{d}x$$

Now add the integral signs:

$$\int \frac{1}{y}\,\mathrm{d}y = \int \mathrm{d}x$$

Integrate each side with respect to the variable:

$$\ln y = x + C$$

Rearrange using the relation between logs and powers:

$$y = \mathrm{e}^{x+C} = \mathrm{e}^{x}\mathrm{e}^{C} = A\mathrm{e}^{x}$$

where $A = \mathrm{e}^{C}$ is a constant. Re-labelling constants is a frequently used trick to make solutions look as uncomplicated as possible.

# Using boundary conditions

The differential equations solved so far have yielded general solutions with a constant of integration. We get particular solutions if we have an boundary condition. These are sometimes called *initial value problems.*

## Using a boundary condition

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6xy^2 \ , \ y(2) = 1$$

Separating the variables gives this:

$$\int \frac{1}{y^2}\,\mathrm{d}y = \int 6x\,\mathrm{d}x$$

Integrating both sides gives:

$$-\frac{1}{y} = 3x^2 + C$$

We can find $C$ by applying the initial condition: $y(2) = 1$ is shorthand for $y = 1$ when $x = 2$.

$$-1 = 3(2^2) + C \ \Rightarrow C = -13$$

Therefore the particular solution is

$$-\frac{1}{y} = 3x^2 - 13$$

Let's rearrange to get $y$:

$$y = \frac{1}{13 - 3x^2}$$

Find the solution to the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y} \ , \ y(3) = 9$$

Separate the variables and include the integral signs:

$$\int y\,\mathrm{d}y = -\int x\,\mathrm{d}x$$

Integrate each side with respect to the variable:

$$\tfrac{1}{2}y^2 = -\tfrac{1}{2}x^2 + C$$

Let's make this easier to deal with before using the initial condition:

$$y^2 = -x^2 + 2C$$

Using $x = 3$ and $y = 9$ we find:

$$81 = -9 + 2C \quad \Rightarrow C = 45$$

The solution the differential equation is

$$y^2 = -x^2 + 90$$

Rearranging we see that

$$x^2 + y^2 = 90$$

So this solution represents a circle centred at the origin with radius $\sqrt{90}$.

# Using the solution

We usually need to make further calculations after solving a differential equation.

Given the differential equation

$$\frac{dN}{dt} = 3N \ , \ N(0) = 10$$

Find the value of $N$ when $t = 0.5$. Give the answer to 3 decimal places.

First — solve by separating the variables:

$$\begin{aligned} \int \frac{1}{N}\, dN &= \int 3\, dt \\ \therefore \ln N &= 3t + C \end{aligned}$$

We're given $N = 10$ when $t = 0$.

$$\therefore \ln 10 = C$$

Particular solution is:

$$\ln N = 3t + \ln 10$$

Use log rules:

$$\begin{aligned} \ln N - \ln 10 &= 3t \\ \ln \frac{N}{10} &= 3t \end{aligned}$$

Remove the logarithm:

$$\frac{N}{10} = e^{3t} \quad \therefore N = 10e^{3t}$$

This is the solution. We'll use it to find $N$ when $t = 0.5$:

$$N = 10e^{3(0.5)} = 44.817 \text{ (3 D.P.)}$$

# Summary / advice

- Differential equations are solved by integration.
- The solution to a differential equation is a function.
- On this course we only study simple (first-order, linear) differential equations that can be solved by separation of variables.
- There is sometimes more than one possible way to separate the variables; different rearrangements should give the same general solution.
- Integration generates constants of integration on both sides of the separated equation; the constants can be merged into a single constant on one side of the equation.
- Try to put the solution in the form $y = f(x)$ but be prepared that in some cases you'll get an implicit function of the form $f(x, y)$.
- Be prepared to re-label constants as you work through. E.g. $2C = A$.
- When you have solved the differential equation you might need to use the solution to calculate something else.

# Test yourself...

Solve the following differential equations. Try to give your answer in the form $y = f(x)$.

1. $\dfrac{1}{(x+4)^2}\dfrac{dy}{dx} = 1$
2. $\dfrac{dy}{dx} = \dfrac{x^3}{y}$
3. $\dfrac{dy}{dx} = \dfrac{y}{x^2}$ given that $y = e^3$ when $x = 1$.
   After solving: find $y$ when $x = 2$ (to 3 DP).
4. $\dfrac{dy}{dx} = 4x^3\cos^2 y$ given that $y = \frac{\pi}{4}$ when $x = 2$.

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Answers:

Constants are denoted by $C$ or $A$. If the answer contains $A$ then a little bit rearranging was necessary after integrating!

1. $y = \frac{1}{3}(x+4)^3 + C$
2. $y = (\frac{1}{2}x^4 + A)^{\frac{1}{2}}$
3. $y = e^{-\frac{1}{x}+4}$ ; $y = 33.115$ when $x = 2$.
4. $y = \tan^{-1}(x^4 - 15)$