INTEGRATION OF PARAMETRIC EQUATIONS

CALCULUS 11

INU0115/515 (MATHS 2)

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Finding the area under a parametric curve

- We saw in the previous presentation how to differentiate parametric equations.
- We can also use integration to find calculate area below curves or volumes of revolution defined by parametric equations.
- The most complicated part of this process is choosing the limits of integration.
- Remember: not paying attention to limits can mean you get the net area rather than the required area. This happens when some of the region is below the x-axis.
- With parametric equations we have the added complication of having to consider the direction in which the curve is traced out as the parameter increases!
- Pay careful attention to the examples in this presentation :-)

Integrating to calculate area

Suppose we want to find the area under a curve defined with the parametric equations

$$x = f(t)\ ,\ y = g(t)$$

We already know that the area under a curve $y = f(x)$ is given by:

$$\text{Area} = \int_a^b y\,\mathrm{d}x$$

where a and b are the limits of integration.

Using the chain rule, the integral to find area becomes:

$$\boxed{\text{Area} = \int_{t_a}^{t_b} y\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t}$$

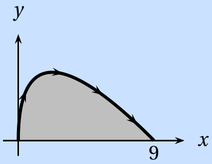
Note that both y and $\frac{\mathrm{d}x}{\mathrm{d}t}$ will be expressed in terms of the parameter t so the integration is with respect to t. We must also make sure to change the limits of integration from x to t where necessary.

Area under a parametric curve

The equations

$$x = t^2 \quad , \quad y = 6t - 2t^2 \quad , t \geq 0$$

define a curve shown below. Find the area of the shaded region.



We need to change the limits from x to t.

$x = 0 \quad \Rightarrow t = 0$

$x = 9 \quad \Rightarrow t = 3.$

Differentiate x with respect to t:

$$\frac{dx}{dt} = 2t$$

The integral for the area is:

$$\text{Area} = \int_0^3 (6t - 2t^2)(2t)\,dt = \int_0^3 (12t^2 - 4t^3)\,dt$$

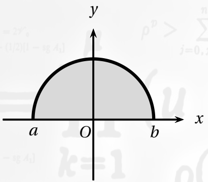
Now integrate with respect to t and apply the limits:

$$\begin{aligned} \text{Area} &= \left[4t^3 - t^4\right]_0^3 \\ &= (108 - 81) - 0 \\ &= 27\,\text{units}^2 \end{aligned}$$

Recap: area and limits

Before we deal with the parametric equations and area any further — let's recap what we know about ordinary integration and area.

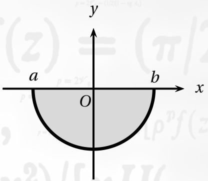
Look at the shaded regions bounded by the curve $y = f(x)$.



This region is above the axis so we integrate from a to b to find the area.

$$\text{Area} = \int_a^b y\,\mathrm{d}x$$

This should be familiar from earlier work!



This region is below the axis so with the same limits we would write

$$\text{Area} = -\int_a^b y\,\mathrm{d}x$$

We include the negative sign to force a positive value from the definite integral — because area doesn't have a sign. Alternatively we could swap the limits.

Consider the parametric equations

$$x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$$

The area integrand is $y\,\mathrm{d}x$:

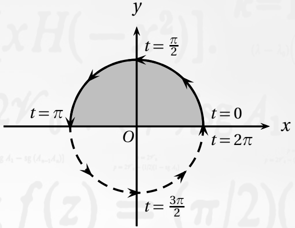
$$\sin t(-\sin t) = -\sin^2 t\,\mathrm{d}t$$

which means we'll use the integral

$$\text{Area} = \int (-\sin^2 t)\,\mathrm{d}t$$

to calculate area.

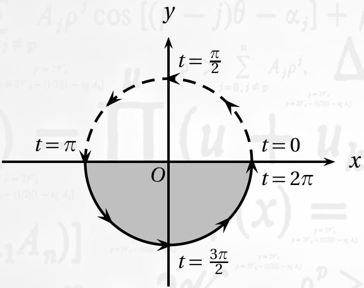
Now as t increases from 0 to 2π the equations trace out a unit circle in the direction indicated by the arrows:



t increases from 0 to π (our lower and upper limits) but x decreases (going from from right to left). We include an extra negative sign in the integral:

$$\begin{aligned} \text{Area} &= -\int_0^{\pi} (-\sin^2 t)\,\mathrm{d}t \\ &= \int_0^{\pi} \sin^2 t\,\mathrm{d}t \\ &= \frac{\pi}{2}\,\text{units}^2 \end{aligned}$$

If we need to find the area beneath the axis, as shown here:



The parameter increases from π to 2π (lower and upper limits). The direction is left to right (no negative sign required) but the area itself is below the axis (negative sign needed). The integral for this is:

$$\begin{aligned} \text{Area} &= -\int_{\pi}^{2\pi} (-\sin^2 t)\,dt \\ &= \int_{\pi}^{2\pi} \sin^2 t\,dt \\ &= \frac{\pi}{2}\,\text{units}^2 \end{aligned}$$

We used 2π (rather than 0) because the area we need is bounded by the lower part of the curve between $\pi \le t \le 2\pi$ (the upper part of the curve is $0 \le t \le \pi$).

Now suppose we want to find the total area enclosed by the circle using integration. It's simply the sum of the two integrals:

$$\begin{aligned}\text{Total area} &= \text{Area}_{above} + \text{Area}_{below}\\ &= \int_0^{\pi} \sin^2 t\,\mathrm{d}t + \int_{\pi}^{2\pi} \sin^2 t\,\mathrm{d}t\\ &= \frac{\pi}{2} + \frac{\pi}{2}\\ &= \pi\end{aligned}$$

One of the properties of integrals is that when upper limit of the first integral is the lower limit of the next, and the function is the same, then we can merge the integrals:

$$\text{Total area} = \int_0^{2\pi} \sin^2 t\,\mathrm{d}t$$

and this also evaluates to give π units^2.

Choosing limits for parametric equations

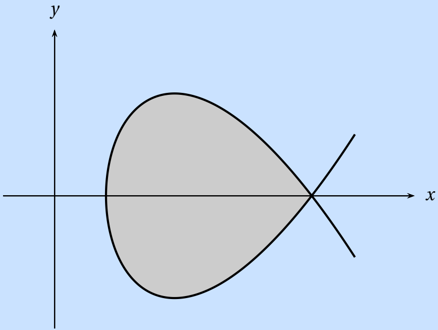
For parametric equations and area we still need to check for negative regions of the graph and also whether x is increasing or decreasing as the parameter increases. The following guidelines should help.

1. Follow the direction in which the parameter is increasing (to get lower and upper limits).
2. Include a negative sign if x is decreasing as t increases (or swap limits).
3. Include a negative sign if the area is below the x-axis.

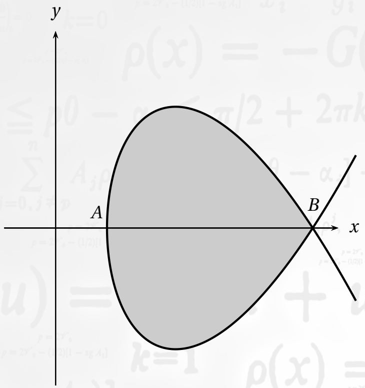
We will keep these ideas in our mind when we integrate with parametric equations. We will study a few examples closely to understand how areas and parametric equations are connected and to keep things simple we'll study a unit circle centred at the origin.

Area beneath a parametric curve

Calculate the shaded area contained in the loop of curve in the picture:



The curve is defined by the parametric equations $x = t^2 + 1$ and $y = t^3 - 4t$.



We need to know the limits of integration (the x-values) at A and B because they define the extent of the loop.

They are on the x-axis, we know that $y = 0$ at those points.

$$t^3 - 4t = 0 \Rightarrow t(t^2 - 4) = 0$$

This tells us that $y = 0$ when $t = 0, \pm 2$.

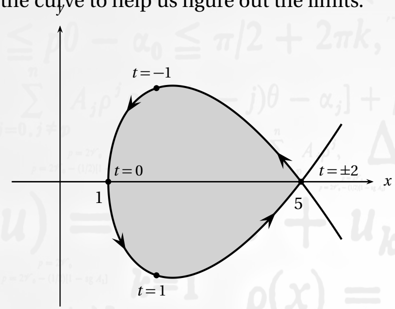
When $t = 0$ this corresponds to $x = 1$.

When $t = 2$ we get $x = 5$ and when $t = -2$ we also find $x = 5$.

So the limits of integration; at A we've got $x = 1$ ($t = 0$). At B we found $x = 5$ — but which way is the curve traced out?

Is our integral going to be $\int_{-2}^{2}$ or $\int_{2}^{-2}$?

It's helpful in this example to mark some additional points (given with values of t) on the curve to help us figure out the limits.



The area integral for this problem will be of the form

$$\int (t^3 - 4t)(2t)\,\mathrm{d}t$$

The upper loop goes from $t = -2$ to 0 (lower to upper limits) but x is decreasing (negative sign needed):

$$\text{Area}_{upper} = -\int_{-2}^{0} (t^3 - 4t)(2t)\,\mathrm{d}t$$

The lower loop goes from $t = 0$ to 2 (lower to upper limits). Although x is increasing, the area is below the axis (so negative sign needed):

$$\text{Area}_{lower} = -\int_{0}^{2} (t^3 - 4t)(2t)\,\mathrm{d}t$$

We can combine these two integrals:

$$\begin{aligned}
\text{Total area} &= -\int_{-2}^{2} 2(t^4 - 4t^2)\,\mathrm{d}t \\
&= -2\left[\tfrac{1}{5}t^5 - \tfrac{4}{3}t^3\right]_{-2}^{2} \\
&= -2\left(\left(\tfrac{32}{5} - \tfrac{32}{3}\right) - \left(-\tfrac{32}{5} + \tfrac{32}{3}\right)\right) \\
&= -2\left(-\tfrac{64}{15}\right) \\
\text{Total area} &= \frac{256}{15}
\end{aligned}$$

So the area is exactly $17\frac{1}{15}$ square units.

Volume of revolution

The volume of revolution obtained when a region underneath the curve $y = f(x)$, bounded by x-axis and the lines $x = a$ and $x = b$ is rotated completely around the x-axis is given by the integral

$$\text{Volume} = \pi \int_a^b y^2 \, \mathrm{d}x$$

As with the area examples, we integrate the parametric equations using substitution.

The limits of the integral are given in terms of $x = a$ and $x = b$ so we must change those to the equivalent limits t_a and t_b as well.

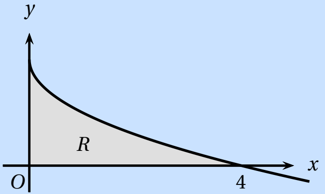
Let's study a simple example.

Volume of revolution

The graph shows the curve defined by the equations

$$x = t^2,\ y = 2 - t,\ t \geq 0$$

Find the volume of revolution obtained when the shaded region is rotated completely around the x-axis.



In terms of x the volume is given by

$$\text{Volume} = \pi \int_0^4 y^2\,\mathrm{d}x$$

Change limits: $x = 0 \Rightarrow t = 0$ and $x = 4 \Rightarrow t = 2$.
Differentiate x:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t \ \Rightarrow \mathrm{d}x = 2t\,\mathrm{d}t$$

Substitute into the volume integral:

$$\text{Volume} = \pi \int_0^2 (2-t)^2 (2t)\,\mathrm{d}t$$

Open the brackets:

$$\begin{aligned} \text{Volume} &= \pi \int_0^2 (4 - 4t + t^2)(2t)\,\mathrm{d}t \\ &= \pi \int_0^2 (8t - 8t^2 + 2t^3)\,\mathrm{d}t \end{aligned}$$

Now integrate:

$$\begin{aligned} \text{Volume} &= \pi \left[4t^2 - \tfrac{8}{3}t^3 + \tfrac{1}{2}t^4\right]_0^2 \\ &= \pi\left[(16 - \tfrac{64}{3} + 8) - (0 - 0 + 0)\right] \\ &= \tfrac{8}{3}\pi \end{aligned}$$

The volume of revolution is $\frac{8}{3}\pi$ units^3.

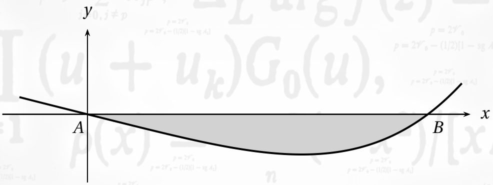
Test yourself...

Solve the following problems using integration.

1. The curve below is defined by the curve

$$x = 4t, \quad y = t^4 - t$$

The shaded region is enclosed by the curve and the x-axis.



Find the t-values at A and B and then use integration to find the exact area of the shaded region.

Answers:

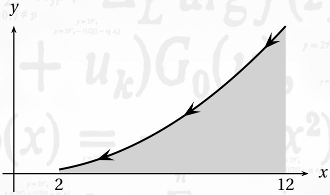
1. $t = 0$ at A and $t = 1$ at B. The total area is $\dfrac{6}{5}$.

Test yourself...

2. The curve below is defined by the curve

$$x = \frac{6}{t}, \; y = \frac{1}{t^2}$$

The shaded region is enclosed by the curve, the x-axis and the lines $x = 2$ and $x = 12$.



Use integration to calculate the exact area of the shaded region.

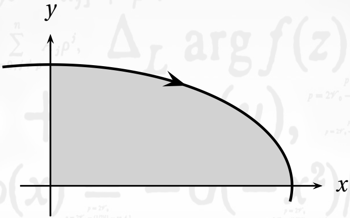
Answers:

2. Area is $\frac{430}{27}$.

Test yourself...

3. The picture below shows part of an ellipse defined by equations

$$x = 4\sin t, \quad y = 2\cos t$$



Use integration to calculate the exact area of the shaded region.

Answers:

3. Area is 2π.