# DIFFERENTIATION OF PARAMETRIC EQUATIONS

## CALCULUS 11

INU0115/515 (MATHS 2)

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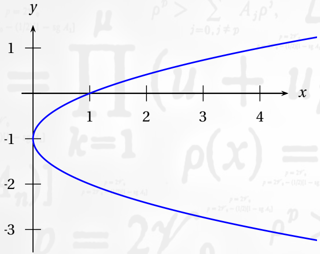
# Introduction

Parametric equations are defined using a parameter. For example

$$x = t^2 \ , \ y = t - 1$$

These equations define a curve in 2D space.




Since it is a curve, we should be able to do all the things we can for cartesian equations, like finding the gradient or calculating the area under the curve.

To do these things we have to differentiate or integrate the equations for the curve.

One possible way of doing these problems is to first change the parametric equations to cartesian form (if possible) and then do the problem like before.

However, this is not necessary because there are methods for differentiating and integrating parametric equations.

# Gradient of a parametric curve

Given a curve defined with the parametric equations

$$x = f(t)\ ,\ y = g(t)$$

We can differentiate each with respect to $t$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f'(t)\ ,\ \frac{\mathrm{d}y}{\mathrm{d}t} = g'(t)$$

The *chain rule* gives usan expression for the gradient:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{1}{\mathrm{d}x/\mathrm{d}t}\end{aligned}$$

Perhaps an easier way to remember the rule is

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{g'(t)}{f'(t)}}$$

The gradient function is also usually a function of the parameter $t$.

## Gradient of a parametric curve

A curve is defined by the parametric equations

$$x = 5 - 2t \,,\; y = 3t^2 + 1$$

Find an expression for the gradient of the curve and its value when $t = 2$.

Differentiate the equations with respect to $t$.

$$\frac{dy}{dt} = 6t \quad \text{and} \quad \frac{dx}{dt} = -2$$

Using the chain rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{6t}{-2} \\ &= -3t \end{aligned}$$

Therefore, at $t = 2$, we find that $\dfrac{dy}{dx} = -6$.

# Finding a tangent equation

Given the parametric equations

$$x = \theta + \cos 3\theta \quad , \quad y = 4 + \sin 2\theta$$

Find the equation of the tangent to the curve at the point where $\theta = 0$.

Differentiate the equations:

$$\frac{dx}{d\theta} = 1 - 3\sin 3\theta \quad , \quad \frac{dy}{d\theta} = 2\cos 2\theta$$

The chain rule gives:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos 2\theta}{1 - 3\sin 3\theta}$$

The value of the gradient and the coordinates of the point on the curve.When $\theta = 0$ we have:

$$\frac{dy}{dx} = \frac{2\cos 0}{1 - 3\sin 0} = 2$$

Therefore the tangent equation has the form $y = 2x + c$. We need a point on the curve to find $c$. When $\theta = 0$:

$$x = 0 + \cos 0 = 1 \quad , \quad y = 4 + \sin 0 = 4$$

So the tangent passes through $(1,4)$. Therefore:

$$4 = 2(1) + c \;\Rightarrow c = 2$$

This means the tangent equation is $y = 2x + 2$.

## Finding a normal equation

A parametric curve is defined by

$$x = 3t^4 - t^2 + 1 \quad , \quad y = 20t$$

Find the equation of a normal line to the point where $t = 1$.

The gradient of the curve is:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{20}{12t^3 - 2t}$$

At the point where $t = 1$, the tangent gradient is

$$m = \frac{dy}{dx} = \frac{20}{12(1)^3 - 2(1)} = 2$$

The gradient of the normal is $-\frac{1}{2}$ (since $m_1 m_2 = -1$).

$$y = -\frac{1}{2}x + c$$

We need a point on the curve to find $c$. When $t = 1$:

$$x = 3(1)^4 - 1^2 + 1 = 3 \quad , \quad y = 20(1) = 20$$

The normal line passes through $(3, 20)$. Therefore:

$$20 = -\frac{1}{2}(3) + c \Rightarrow c = \frac{43}{2}$$

The normal equation is $y = -\frac{1}{2}x + \frac{43}{2}$.

# Test yourself...

1. Show that the parametric equations
$$x = 1 - t^2,\ y = 4t^2$$
represent a straight line.
2. Given the parametric equations
$$x = \frac{1}{t},\ y = 3t$$
Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $t$, and also find the Cartesian equation.
3. Given the parametric equations
$$x = \sec t,\ y = t$$
find the normal equation to the point where $t = \frac{\pi}{4}$.

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Answers:

1. $\frac{\mathrm{d}y}{\mathrm{d}x} = -4$ (constant gradient; therefore a line).
2. $\frac{\mathrm{d}y}{\mathrm{d}x} = -3t^2$. Cartesian equation: $xy = 3$.
3. $y = -\sqrt{2}x + \frac{\pi}{4} + 2$