PARAMETRIC EQUATIONS

CALCULUS 11

INU0115/515 (MATHS 2)

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Introduction

Cartesian

The relationship between two variables such as x and y is represented explicitly, such as

$$y = x^2 - 5x + 6$$

or implicitly like

$$xy^2 = 2\sin x$$

Parametric

The relationship between two variables is represented via third variable, called a parameter. Here are some *parametric equations*:

$$x = t + 1 \, , \; y = \frac{1}{2}t^2$$

Values of x and y are calculated by choosing values for the parameter t.

We will examine the general case for two parametric equations:

$$x = f(t) \, , \; y = g(t)$$

Were $f(t)$ and $g(t)$ are functions of the parameter t. The parameter may take on all possible values or its value may be restricted to some interval. Each value of t specifies a point (x, y) in the plane and so the parametric equations can represent curves or lines.

Plotting parametric equations

A simple procedure for plotting graphs defined by parametric equations is to calculate coordinates (x, y) and plot them.

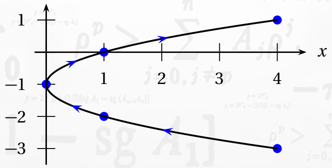
For example, given the equations

$$x = t^2\ ,\ y = t - 1$$

We can make a table of values for the parameter t and calculate the coordinates for x and y.

t	−2	−1	0	1	2
x	4	1	0	1	4
y	−3	−2	−1	0	1

Now plot coordinates (x, y) and draw a smooth curve through the points.



Arrows indicate how the curve is traced by increasing the parameter.

Plotting a parametric curve

Plot the curve defined by the parametric equations

$$x = \cos t\ ,\ \ y = \sin t\ ,\ 0 \leq t \leq 360^\circ$$

We can construct a table of values for selected values of the parameter t and plot as points the corresponding values of (x, y). As we are sketching the points, we'll use one decimal place accuracy.

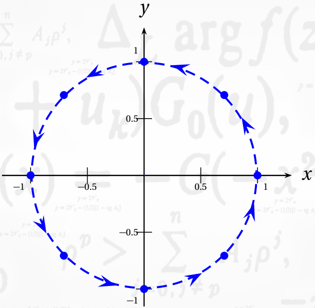
t	0°	45°	90°	135°	180°	225°	270°	315°	360°
x	1	0.7	0	−0.7	−1	−0.7	0	0.7	1
y	0	0.7	1	0.7	0	−0.7	−1	−0.7	0

To see the curve given by the parametric equations, we can plot the coordinates (x, y) on a graph.

Plotting a parametric curve

Sketch the curve defined by the parametric equations

$$x = \cos t\ ,\ \ y = \sin t\ ,\ 0 \le t \le 360^\circ$$



By plotting the points (x, y) from the table and drawing a smooth curve through them we get a circle centred on the origin. Plotting more and more coordinates will verify this.

But can we *prove* that these parametric equations really represent a circle?

Changing from parametric to Cartesian form

Parametric equations can be changed to Cartesian form by eliminating the parameter. The procedure for doing this may involve simple substitution or a more complicated use of trig identities.

Let's consider a simple case first. Given the parametric equations

$$x = t + 1\,,\ y = t^2 - 2$$

Rearrange the first to obtain t:

$$t = x - 1$$

Substitute this into the second equation:

$$y = (x-1)^2 - 2$$

The parameter is eliminated. We can expand the brackets (if we want to!) and simplify:

$$y = x^2 - 2x - 1$$

So the parametric equations actually represented a quadratic curve.

Changing from parametric to Cartesian form

Express the curve defined by

$$x = 2t^2 \,,\; y = 4t$$

in Cartesian form.

Rearrange the equation for y to get t

$$t = \frac{y}{4}$$

Substitute into the equation for x:

$$x = 2\left(\frac{y}{4}\right)^2 = \frac{y^2}{8}$$

We can write this $y^2 = 8x$ (this is a quadratic with the x and y swapped over from the more usual form. Can you sketch this?)

Converting with trig identities

Equations with trig functions need to be treated differently in order to eliminate the parameter.

The following Pythagorean identities will prove to be useful in many cases so try to remember them!

$$\begin{aligned} \sin^2\theta + \cos^2\theta &\equiv 1 \\ 1 + \tan^2\theta &\equiv \sec^2\theta \\ 1 + \cot^2\theta &\equiv \operatorname{cosec}^2\theta \end{aligned}$$

Let's look at a few examples next...

Changing from parametric to Cartesian form

Change parametric equations

$$x = \cos t\,,\ \ y = \sin t\ ,\ 0 \leq t \leq 360^\circ$$

into Cartesian form.

The identity $\sin^2 t + \cos^2 t \equiv 1$ connects the two equations.

Squaring and adding the parametric equations gives:

$$x^2 + y^2 = \cos^2 t + \sin^2 t$$

But the RHS is identitical to 1, so that:

$$x^2 + y^2 = 1$$

This is the equation of a circle centred at the origin.

Changing from parametric to Cartesian form

Change parametric equations

$$x = 3\sin\theta\ ,\ \ y = 4\cos\theta\ ,\ 0 \le \theta \le 360^\circ$$

into Cartesian form.

First, rearrange each equation to get $\sin\theta$ and $\cos\theta$:

$$\sin\theta = \frac{x}{3}\ ,\ \cos\theta = \frac{y}{4}$$

Now square and add these equations:

$$\sin^2\theta + \cos^2\theta = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2$$

But $\sin^2\theta + \cos^2\theta \equiv 1$ so this equation becomes:

$$\begin{aligned} \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 &= 1 \\ \frac{x^2}{9} + \frac{y^2}{16} &= 1 \end{aligned}$$

This equation describes an ellipse.

Changing parametric equations to Cartesian form

A curve is described by the parametric equations

$$x = 4\sec\theta \ , \ y = 2\tan\theta$$

Give the curve in Cartesian form.

Use the identity $\sec^2\theta \equiv 1 + \tan^2\theta$.

So square both parametric equations:

$$x^2 = 16\sec^2\theta \ , \ y^2 = 4\tan^2\theta$$

Rearrange to get:

$$\frac{x^2}{16} = \sec^2\theta \ , \ \frac{y^2}{4} = \tan^2\theta$$

Substitute into the identity:

$$\frac{x^2}{16} = 1 + \frac{y^2}{4}$$

This is usually written in the same form as a circle equation:

$$\frac{x^2}{16} - \frac{y^2}{4} = 1$$

This equation represents a curve called a hyperbola.

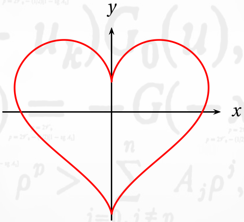
Why use parametric equations?

Parametric equations are sometimes preferred because they can describe complex curves more easily than the equivalent Cartesian equation.

For example, consider the parametric equations

$$x = 12\sin t - 4\sin 3t \ , \ \ y = 13\cos t - 5\cos 2t - 2\cos 3t - \cos 4t$$

where $-\pi \le t \le \pi$ and which defines this curve:



then the equivalent implicit Cartesian equation is:

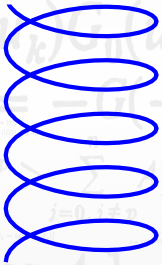
$$\begin{aligned} x^8 + 240x^6y + 8116x^6 + 256x^4y^3 + 70464x^4y^2 + 597312x^4y + 15918304x^4 \\ + 67584x^2y^4 + 1818624x^2y^3 + 6081792x^2y^2 - 471039744x^2y - 3937380544x^2 \\ + 16384y^6 + 589824y^5 + 2899968y^4 - 71958528y^3 - 246497280y^2 + 4261478400y - 10061824000 = 0 \end{aligned}$$

3D parametric curves

Parametric equations can describe curves in 3D space efficiently. For example the following set of equations:

$$x = \cos t\ ,\ y = \sin t\ ,\ z = t$$

describe a curve called a helix.



It is not practical to express such curves in Cartesian form! But with calculus it is possible to calculate the lengths and other properties of them.

Solving problems with parametric equations

We'll spend some time learning to manipulate parametric equations to solve simple problems.

Intersection with the x-axis

Given the parametric equations

$$x = t - 1 \ , \ y = 4 - t^2$$

Find the coordinates of the points where the curve meets the x-axis.

We are required to find the places where $y = 0$, so

$$\begin{aligned} 4 - t^2 &= 0 \\ 4 &= t^2 \\ \therefore t &= \pm 2 \end{aligned}$$

When $t = 2$ then $x = 1$ and when $t = -2$ then $x = -3$. So the curve meets the x-axis at

$$(-3, 0) \text{ and } (1, 0)$$

Intersection of a curve and a line

A curve has parametric equations given by

$$x = 2t + 1 \ , \ y = 1 - t$$

Find the point of intersection of the curve with the line $y = x + 1$.

Substitute the parametric equations into the Cartesian equation for the line.

$$\begin{aligned} y &= x + 1 \\ 1 - t &= (2t + 1) + 1 \end{aligned}$$

Rearrange to solve for t:

$$t = -\tfrac{1}{3}$$

Substitute the parameter into the parametric equations:

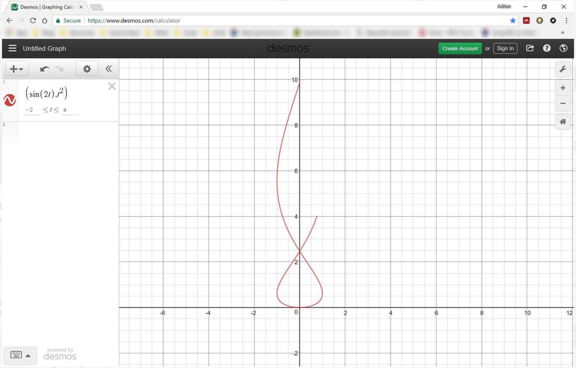
$$x = 2(-\tfrac{1}{3}) + 1 = \tfrac{1}{3} \ , \ y = 1 - (-\tfrac{1}{3}) = \tfrac{4}{3}$$

So the intersection point is $\left(\frac{1}{3}, \frac{4}{3}\right)$.

Desmos

Parametric equations can be plotted in Desmos (see `www.desmos.com/calculator`).

Equations are entered in the format of coordinates $(x(t), y(t))$.



The picture shows the graph of

$$x = \sin 2t \qquad y = t^2 \qquad -2 \le t \le \pi$$

Constants such as π are typed as "pi".

Test yourself...

Let's practice what we've learned about parametric equations.

1. Express $x=4t-3$, $y=t^3$ in Cartesian form.
2. Express $x=2\sec t$, $y=\tan^2 t$ in Cartesian form.
3. Express $x=1+\cot\theta$, $y=3\sin\theta$ in Cartesian form
4. Find where the curve

$$x=t+1\,,\; y=t^2,\;\; t\geq 0$$

intersects the line $y=2x+1$.

Answers:

1. $y=\left(\dfrac{x+3}{4}\right)^3$
2. $y=\left(\dfrac{x}{2}\right)^2-1$
3. $y^2=\dfrac{9}{x^2-2x+2}$
4. $(4,9)$