Mean and r.m.s. values

Calculus 10

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Introduction

You are probably familiar with how to calculate the mean (average) value of a list of numbers.

For example, what is the mean value of the integers $1, 2, \ldots, 10$?

$$\text{Mean value} = \frac{1 + 2 + \ldots + 10}{10} = \frac{55}{10} = 5.5$$

The purpose of a mean value is to represent the values of all the others in the calculation. For a set of discrete values like the numbers 1 to 10, the calculation is easy.

Scientists often deal with quantities or functions which vary smoothly and continuously - such as speed or temperature. For example in cases where an electric current is changing with time, it would still be useful to calculate an average value for the current. The simple method for mean value used above cannot easily be applied to such quantities.

Fortunately we can use integration instead.

Mean value of a function

The mean value of a function $f(x)$ between $x = a$ and $x = b$ is

$$\boxed{\text{Mean Value} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x}$$

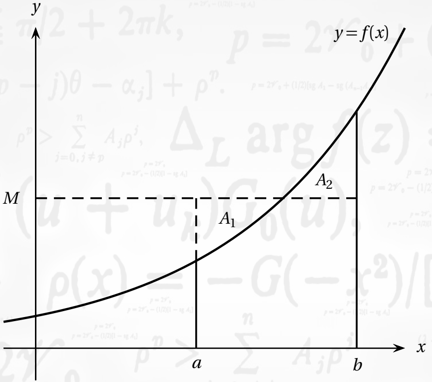
Rearranging this we get an interesting relationship between the quantities:

$$(b-a) \times (\text{Mean Value}) = \int_a^b f(x)\,\mathrm{d}x$$

The RHS represents the area between the curve and the x-axis between $x = a$ and $x = b$. The LHS represents a rectangle with base $b-a$ and a height equal to the mean value.

A picture of this is shown on the next slide.

Picture of the mean value of a function



The mean value M is calculated so that the areas A_1 and A_2 above and below the line $y = M$ and between the curve in the interval (a, b), are equal.

Also, the area of the rectangle with base $b - a$ and height M is equal to the area under the curve over the same interval.

Mean value of a function

Calculate the mean value of the function $y = e^x$ over the interval $x = 0$ to $x = 3$.

$$\begin{aligned} \text{Mean Value} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{1}{3-0}\int_0^3 e^x\,dx \\ &= \frac{1}{3}\left[e^x\right]_0^3 \\ &= \frac{1}{3}\left[(e^3 - e^0\right] \\ &= \tfrac{1}{3}(e^3 - 1) \end{aligned}$$

So the mean (average) value of the function over this interval is 6.362 (to 3 decimal places).

Mean value of a function

Find the mean value of $y = \sin x$ over the interval $0 \le x \le \pi$.

$$\begin{aligned}
\text{Mean Value} &= \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\pi \sin x\,\mathrm{d}x \\
&= -\frac{1}{\pi}[\cos x]_0^\pi \\
&= -\frac{1}{\pi}[(\cos\pi - \cos 0] \\
&= \frac{2}{\pi}
\end{aligned}$$

So the mean (average) value of the function over this interval is 0.637 (to 3 decimal places).

What would be the mean value of this function over $0 \le x \le 2\pi$?

R.M.S. values

The r.m.s. value of a function is the square-root of the mean value of the squares of the function between some given limits. R.M.S. is short for root mean square. It is also called the quadratic mean.

We can write this definition as

$$\boxed{(\text{R.M.S.})^2 = \frac{1}{b-a}\int_a^b y^2\,\mathrm{d}x}$$

R.M.S. values are useful in electrical engineering where measurements can take positive and negative values (e.g. alternating electric currents).

Calculation of r.m.s.

Find the r.m.s. value of $y = x^2 + 3$ between $x = 1$ and $x = 3$.

$$
\begin{aligned}
(\text{r.m.s.})^2 &= \frac{1}{3-1}\int_1^3 (x^2+3)^2\,\mathrm{d}x \\
&= \tfrac{1}{2}\int_1^3 (x^4+6x^2+9)\,\mathrm{d}x \\
&= \tfrac{1}{2}\left[\tfrac{1}{5}x^5+2x^3+9x\right]_1^3 \\
&= \tfrac{1}{2}\times\frac{592}{5} \\
&= \frac{592}{10} \\
\text{r.m.s} &= \sqrt{\frac{592}{10}}
\end{aligned}
$$

So the r.m.s. value of the functon is 7.69 (to 2 decimal places).

Mean value of a function

Calculate the r.m.s. value (to 3DP) for $y = \sin x$ over the interval $0 \le x \le 2\pi$.

$$(\text{r.m.s.})^2 = \frac{1}{b-a}\int_a^b y^2\,\mathrm{d}x = \tfrac{1}{2\pi}\int_0^{2\pi} \sin^2 x\,\mathrm{d}x$$

This integral is solved using the double angle formula:

$$\begin{aligned}
&= \tfrac{1}{2\pi}\int_0^{2\pi} \tfrac{1}{2}(1-\cos 2x)\,\mathrm{d}x \\
&= \tfrac{1}{4\pi}\int_0^{2\pi} (1-\cos 2x)\,\mathrm{d}x \\
&= \tfrac{1}{4\pi}\left[x - \tfrac{1}{2}\sin 2x\right]_0^{2\pi} \\
&= \tfrac{1}{4\pi}\left(2\pi - \tfrac{1}{2}\sin 2\pi\right) \\
&= \tfrac{1}{4\pi}\times 2\pi \\
(\text{r.m.s.})^2 &= \tfrac{1}{2} \\
\text{r.m.s.} &= \tfrac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.707
\end{aligned}$$

Summary

Average values for a function $y = f(x)$ over an interval $a \leq x \leq b$ can be calculated in a number of ways.

Two of the most commonly used are:

$$\boxed{\text{Mean Value } = \frac{1}{b-a}\int_a^b y\,\mathrm{d}x}$$

and

$$\boxed{(\text{R.M.S.})^2 = \frac{1}{b-a}\int_a^b y^2\,\mathrm{d}x}$$

The r.m.s. value is more appropriate for oscillating functions (e.g. sine or cosine based curves).

Test yourself...

Let's practice calculating mean and r.m.s. values now...

1. Calculate the mean value of $f(x) = x^6$ over the interval $0 \le x \le 1$.
2. Calculate the r.m.s. value of $f(x) = \sqrt{x \sin x}$ over the interval $0 \le x \le \pi$.
3. Calculate the mean value of $f(x) = 2^x$ over the interval $-1 \le x \le 2$.
4. Calculate the r.m.s. value of $f(x) = 4\cos 2x$ over the interval $-\pi \le x \le \pi$.

Answers:

1. Mean value is $\frac{1}{7}$
2. r.m.s. value is 1.
3. Mean value is $\dfrac{7}{6\ln 2}$.
4. r.m.s. value is $\sqrt{8}$.