Volume of revolution

Calculus 10

INU0115/515 (Maths 2)

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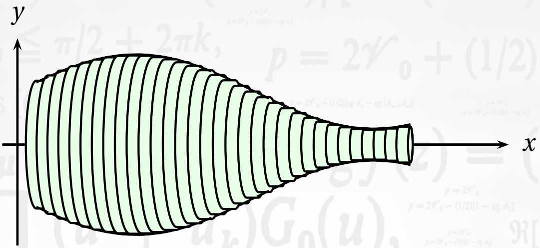
Solid of revolution



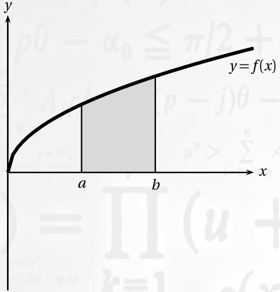
Rotating a curve completely around the x-axis produces a 3D shape.

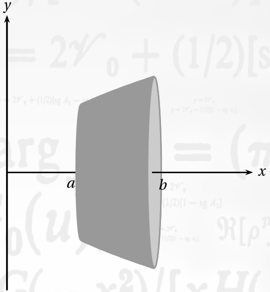
When a curve is rotated (or revolved) around the x-axis or y-axis it will trace out an area which encloses a volume of space. This 3D shape is called a solid of revolution.

We can use integration techniques to calculate the exact volume of the solid of revolution.

On this course we'll see how to calculate the so-called volume of revolution for curves rotated around the x-axis.

Volume of revolution





Consider the curve of $y = f(x)$ shown above. We are interested in the section of curve bounded by the lines $x = a$ and $x = b$.

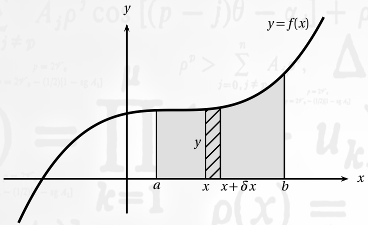
Think about the 3D shape obtained if the shaded region under the curve is rotated completely around the x-axis.

Rotating the shaded region through 360° (or 2π rad) generates a 3D shape called the volume of revolution.

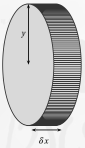
On the following slides we will derive a method for calculating the volume of this solid.

Deriving the volume of revolution

Consider the curve $y = f(x)$ and a narrow strip of width δx and height y somewhere between the lines $x = a$ and $x = b$.



Rotating the narrow strip around the x-axis generates narrow disk of volume δV:



The volume of this disk is found using the formula for a cylinder and is given by

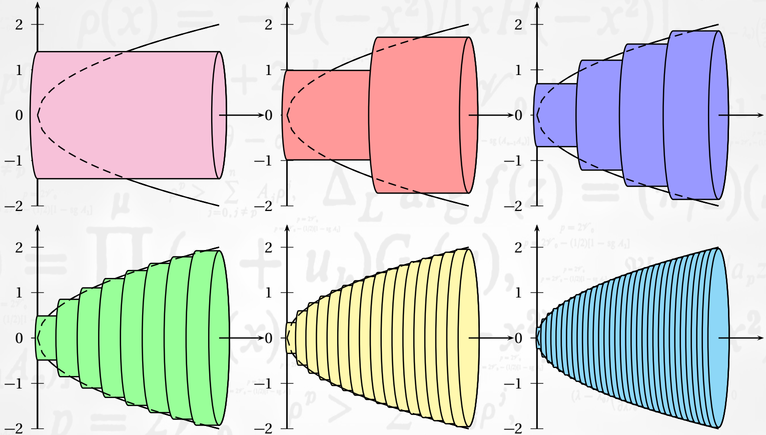
$$\delta V = \pi y^2 \delta x$$

The total volume of revolution is estimated by partitioning the region into many disks like this one. The total volume is approximately that of the sum of the volumes of these narrow disks.

$$\text{Volume} \approx \sum_{x=a}^{x=b} \delta V \approx \sum_{x=a}^{x=b} \pi y^2 \delta x$$

If we let $\delta x \to 0$ then this quantity can be shown to be the same as the definite integral

$$\boxed{\text{Volume} = \pi \int_a^b y^2 \, \mathrm{d}x}$$

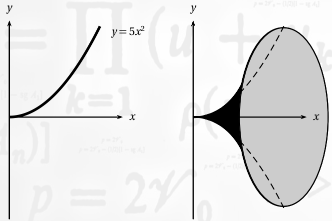


The volume of revolution can be approximated by taking an increasing number of thin disks of width δx. In the limit $\delta x \to 0$ the volume can be found by integration.

Calculating a volume of revolution

Calculate the volume generated when the graph of $y=5x^2$ between the origin and the line $x=2$ is rotated completely around the x-axis.

A sketch of the function and the volume of revolution is shown below:



Using volume formula we substitute the function $y=5x^2$ and the limits $x=0$ and $x=2$ into it:

$$\begin{aligned}\text{Volume} &= \pi\int_0^2 (5x^2)^2\,\mathrm{d}x\\ &= \pi\int_0^2 25x^4\,\mathrm{d}x\\ &= \pi\left[5x^5\right]_0^2\\ &= 5\pi\left[x^5\right]_0^2\\ &= 5\pi\left(2^5-0\right)\\ &= 160\pi\end{aligned}$$

To an accuracy of 3 significant figures the volume is 503 units^3.

Calculating a volume of revolution

Calculate the volume of revolution obtained when the curve $f(x) = \mathrm{e}^x + 1$ between the lines $x = 0$ and $x = 1$ is rotated completely around the x-axis.

The volume is obtained by putting the function and limits into the volume formula.

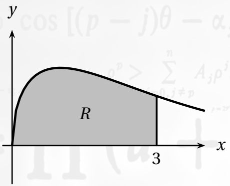
$$\text{Volume} = \int_a^b \pi y^2 \,\mathrm{d}x = \pi \int_0^1 (\mathrm{e}^x + 1)^2 \,\mathrm{d}x$$

Expand the brackets and integrate term by term:

$$\begin{aligned}
\text{Volume} &= \pi \int_0^1 (\mathrm{e}^{2x} + 2\mathrm{e}^x + 1)\,\mathrm{d}x \\
&= \pi \left[\tfrac{1}{2}\mathrm{e}^{2x} + 2\mathrm{e}^x + x \right]_0^1 \\
&= \pi \left[\left(\tfrac{1}{2}\mathrm{e}^2 + 2\mathrm{e} + 1 \right) - \left(\tfrac{1}{2} + 2 + 0 \right) \right] \\
&= \pi \left(\tfrac{1}{2}\mathrm{e}^2 + 2\mathrm{e} + 1 - \tfrac{1}{2} - 2 \right) \\
&= \pi \left(\tfrac{1}{2}\mathrm{e}^2 + 2\mathrm{e} - \tfrac{3}{2} \right) \\
\therefore \text{Volume} &= \tfrac{1}{2}\pi \left(\mathrm{e}^2 + 4\mathrm{e} - 3 \right) \text{ units}^3
\end{aligned}$$

A more complicated volume problem

The picture shows a region R enclosed by the x-axis and the curve $y = \sqrt{x}e^{-\frac{1}{2}x}$ and the line $x = 3$. Find the volume of revolution produced when R is rotated by 2π radians around the x-axis, to 3 significant figures.



The volume of revolution in this case is obtained from:

$$\text{Volume} = \pi \int_0^3 \left(\sqrt{x}e^{-\frac{1}{2}x}\right)^2 \, dx$$
$$= \pi \int_0^3 xe^{-x} \, dx$$

Use integration by parts. Let:

$$u = x \quad \text{and} \quad dv = e^{-x}$$

Therefore $du = 1$ and $v = -e^{-x}$.

The volume integral is:

$$\text{Volume} = \pi \left[-xe^{-x} - \int (-e^{-x})(1) \, dx \right]_0^3$$
$$= \pi \left[-xe^{-x} + \int e^{-x} \, dx \right]_0^3$$
$$= \pi \left[-xe^{-x} - e^{-x} \right]_0^3$$
$$= \pi \left[-e^{-x}(x+1) \right]_0^3$$

Evaluating the limits gives:

$$\text{Volume} = \pi \left((-4e^{-3}) - (-e^{0}) \right)$$
$$= (1 - 4e^{-3})\pi \approx 2.52 \, \text{units}^3$$

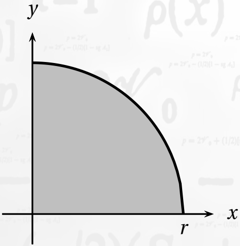
An exam question

Write down the equation of of the circle with centre (0,0) and radius r. By rotating the region bounded by the coordinate axes and the part of the circle in the first quadrant about the x-axis, show that the volume of a hemisphere is $\frac{2}{3}\pi r^3$.

The equation of the circle centred at (0,0) of radius r is

$$x^2+y^2=r^2$$

A sketch showing the region to be rotated:



The volume of the rotated region is

$$\begin{aligned}\text{Volume} &= \pi\int_0^r y^2\,\mathrm{d}x \\ &= \pi\int_0^r (r^2-x^2)\,\mathrm{d}x\end{aligned}$$

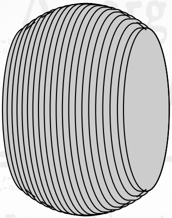
Integrate and evaluate:

$$\begin{aligned} &= \pi\left[r^2x-\tfrac{1}{3}x^3\right]_0^r \\ &= \pi\left(r^2.r-\tfrac{1}{3}r^3\right)-0 \\ &= \pi\left(r^3-\tfrac{1}{3}r^3\right) \\ \therefore \text{Volume} &= \tfrac{2}{3}\pi r^3\end{aligned}$$

Test yourself...

Let's practice some volume problems.

1. The shape below was obtained by rotating the curve $y = 4 - x^2$, $-1 \leq x \leq 1$ completely around the x-axis.



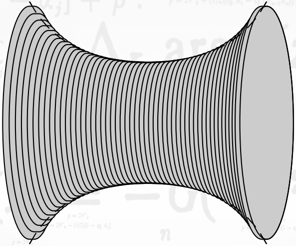
Use integration to calculate the exact volume of this shape.

Answers:

1. $V = \dfrac{406\pi}{15}$ units^3

Test yourself...

2. The shape below was obtained by rotating the curve $y = 3\sec x$, $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$ completely around the x-axis.



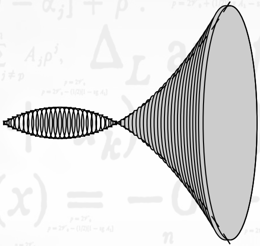
Use integration to calculate the exact volume of this shape.

Answers:

2. $V = 18\pi\sqrt{3}\,\text{units}^3$

Test yourself...

3. The shape below was obtained by rotating the curve $y = x(x-1)$, $0 \le x \le 2$ completely around the x-axis.



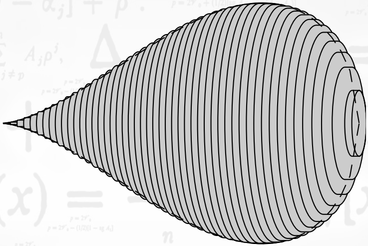
Use integration to calculate the exact volume of this shape.

Answers:

3. $V = \frac{16}{15}\pi\,\text{units}^3$

Test yourself...

4. The shape below was obtained by rotating the curve $y = x\sqrt{\sin x}$, $0 \le x \le \pi$ completely around the x-axis.



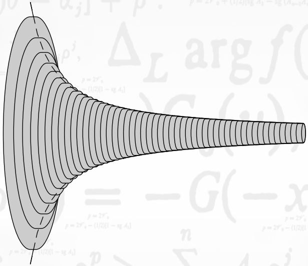
Use integration to calculate the exact volume of this shape.

Answers:

4. $V = \pi(\pi^2 - 4)\,\text{units}^3$

Test yourself...

5. The shape below, known as Gabriel's Horn, was obtained by rotating the curve $y = \frac{1}{x}$, $a \leq x \leq \infty$ completely around the x-axis, where a is a positive constant.



The Horn is infinitely long. Use integration to show that volume is actually finite.

Answers:

5. $V = \frac{\pi}{a}$ units^3